Math-3
Lesson 5-2

Radicals and Rational Exponents

$\sqrt{3}$ What number is equivalent to the square root of 3?

$x = \sqrt{3}$ Square both sides of the equation

$(x)^2 = \left(\sqrt{3}\right)^2$ $x^2 = 3$

$x = \sqrt{3}$ is an equivalent statement to $x^2 = 3$

$\sqrt{3}$ ≈ 1.732

≈ 1.7321

≈ 1.73205

≈ 1.732051

$\approx 1.7320508...$

There is no equivalent number

The decimal, is just an approximation.

Radicals

Index number

Radical symbol

$$\sqrt{3} \rightarrow \sqrt[2]{3}$$

Radicand

$$x = \sqrt[2]{3}$$

$$x^2 = 3$$

The "square root of 3" means:
"what number squared equals 3?"

$$x = \sqrt[3]{4}$$

$$x^3 = 4$$

The "3rd root of 4" means:
"what number cubed equals 4?"

Adding and subtracting radicals

Can these two terms be combined using addition? $3x+2x$

Write 3x as repeated addition $x+x+x$

Write 2x as repeated addition $x+x$

$$3x+2x \rightarrow x+x+x+x+x \rightarrow 5x$$

When multiplication is written as repeated addition, "like terms" look exactly alike.

$$3\sqrt{x}+2\sqrt{x} \rightarrow \sqrt{x}+\sqrt{x}+\sqrt{x}+\sqrt{x}+\sqrt{x} \rightarrow 5\sqrt{x}$$

$$3\sqrt{6}+2\sqrt{6} \rightarrow \sqrt{6}+\sqrt{6}+\sqrt{6}+\sqrt{6}+\sqrt{6} \rightarrow 5\sqrt{6}$$

Define “like powers” “Same base, same exponent”.

$$3x^4 + 2x^4 \rightarrow 5x^4$$

Define “like radicals” “Same radicand, same index number”.

$$3\sqrt{6} + 2\sqrt{6} \rightarrow 5\sqrt{6}$$

Which of the following are “like radicals” that can be added?

$\sqrt{2} + \sqrt{3}$ no

$2\sqrt{3} + 3\sqrt{2}$ no

$\sqrt[4]{2} + \sqrt[3]{2}$ no

$\sqrt[4]{5} + \sqrt[4]{5}$ yes

$3\sqrt[5]{2} + 4\sqrt[5]{2}$ yes

$6\sqrt[3]{4} + 6\sqrt[4]{4}$ yes

$\sqrt{3}+\sqrt{2} \rightarrow \sqrt{3+2}=\sqrt{5}$ Are they equivalent?

If this is a property of radicals, it must work for every combination of numbers.

$$\sqrt{4}+\sqrt{9} \rightarrow \sqrt{13}$$

$$\sqrt{4}+\sqrt{9} \rightarrow 2+3 \rightarrow 5 \neq \sqrt{13}$$

$$\sqrt{a}+\sqrt{b} \neq \sqrt{a+b}$$

This is NOT a property of radicals.
NEVER DO THIS!!!!

$$\sqrt{4} * \sqrt{9} \rightarrow \sqrt{4 * 9}$$

Are these equivalent?

$$\sqrt{a} * \sqrt{b} \rightarrow \sqrt{a * b}$$

$$2 * 3 \rightarrow \sqrt{36}$$

$$2 * 3 \rightarrow 6$$

$$6 = 6$$

Yes!

<u>Product of Radicals Property</u>

$$\sqrt{a} * \sqrt{b} = \sqrt{ab}$$

$$\sqrt{5} * \sqrt{2} = \sqrt{10}$$

$$\sqrt{a} * \sqrt{b} = \sqrt{ab}$$

Simplify the following:

$$3\sqrt{8} * 5\sqrt{2}$$

$$3 * \sqrt{8} * 5 * \sqrt{2}$$

$$3 * 5 * \sqrt{8} * \sqrt{2}$$

$$15 * \sqrt{8} * \sqrt{2}$$

$$15 * \sqrt{16}$$

$$15 * 4 = 60$$

$$2\sqrt{3} * 3\sqrt{5} \rightarrow 6\sqrt{15}$$

$$7\sqrt{6} * 2\sqrt{5} \rightarrow 14\sqrt{30}$$

$$\sqrt{5} + 3\sqrt{5} \rightarrow 4\sqrt{5}$$

$$7\sqrt{6} + 2\sqrt{6} \rightarrow 9\sqrt{6}$$

<u>Simplify radicals</u>: use the Product of Radicals Property to factor ("break apart") the radical into a "perfect square" times a number.

$$\sqrt{a} * \sqrt{b} = \sqrt{ab}$$

$$\sqrt{18} \rightarrow \sqrt{9} * \sqrt{2} \rightarrow 3 * \sqrt{2} \rightarrow 3\sqrt{2}$$

<u>Simplify</u>

$$\sqrt{24} \rightarrow \sqrt{4} * \sqrt{6} \rightarrow 2\sqrt{6}$$

$$3\sqrt{32x^2} \rightarrow 3 * \sqrt{16} * \sqrt{x^2} + \sqrt{2} \rightarrow 3 * 4 * x * \sqrt{2} \rightarrow 12x\sqrt{2}$$

$$\sqrt[3]{x^4} \rightarrow \sqrt[3]{x^3} * \sqrt[3]{x} \rightarrow x\sqrt[3]{x}$$

$$\sqrt[4]{3x^5y} \rightarrow \sqrt[4]{x^4} * \sqrt[4]{3xy} \rightarrow x\sqrt[4]{3xy}$$

Can we add "unlike" radicals?

$$\sqrt{a} * \sqrt{b} = \sqrt{ab}$$

Simplify

$$7\sqrt{6} + 2\sqrt{24} \rightarrow 7\sqrt{6} + (2 * \sqrt{4} * \sqrt{6})$$

$$\rightarrow 7\sqrt{6} + (2 * 2 * \sqrt{6})$$

$$\rightarrow 7\sqrt{6} + 4\sqrt{6}$$

$$\rightarrow 11\sqrt{6}$$

$$-3\sqrt{32} + 2\sqrt{8} \rightarrow (-3 * \sqrt{16} * \sqrt{2}) + (2 * \sqrt{4} * \sqrt{2})$$

$$\rightarrow (-3 * 4 * \sqrt{2}) + (2 * 2 * \sqrt{2})$$

$$\rightarrow -12\sqrt{2} + 4\sqrt{2}$$

$$\rightarrow -8\sqrt{2}$$

Another way to Simplify Radicals

Factor, factor, factor!!!

$$\sqrt{54} \rightarrow \sqrt[2]{54} \rightarrow \sqrt[2]{2*27} \rightarrow \sqrt[2]{2*3*9} \rightarrow \sqrt[2]{2*3*3*3}$$

What is the factor that is used (Index number) '2' times under the radical?

Bring the out factor that is used 2 times.

$$\rightarrow 3\sqrt[2]{2*3} \quad \rightarrow 3\sqrt{6}$$

Using Properties of Exponents to reduce the writing:

$$\sqrt[4]{32x^6} \rightarrow \sqrt[4]{32*x^4*x^2}$$

$$\rightarrow x\sqrt[4]{32*x^2}$$

$$\rightarrow x\sqrt[4]{2^4*2^1*x^2}$$

$$\rightarrow 2x\sqrt[4]{2x^2}$$

Rationalizing the denominator: using mathematical properties to change an irrational number (or imaginary number) in the denominator into a rational number.

We take advantage of the idea:

$$\sqrt{2} * \sqrt{2} = \sqrt{2*2} = \sqrt{4} = 2$$

$$\sqrt{3} * \sqrt{3} = \sqrt{3*3} = \sqrt{9} = 3$$

$$\frac{1}{\sqrt{2}} * \frac{\sqrt{2}}{\sqrt{2}} \rightarrow \frac{\sqrt{2}}{2}$$

Identity Property of Multiplication

multiplying by '1' doesn't change the number.

$$\frac{2}{\sqrt{6}} * \frac{\sqrt{6}}{\sqrt{6}} \rightarrow \frac{2\sqrt{6}}{6} \rightarrow \frac{2*\sqrt{6}}{2*3} \rightarrow \frac{\sqrt{6}}{3}$$

$$\frac{25}{\sqrt{15}} * \frac{\sqrt{15}}{\sqrt{15}} \rightarrow \frac{25\sqrt{15}}{15} \rightarrow \frac{5*5**\sqrt{15}}{5*3} \rightarrow \frac{5\sqrt{15}}{3}$$

$$\frac{14}{3\sqrt{21}} * \frac{\sqrt{21}}{\sqrt{21}} \rightarrow \frac{14\sqrt{21}}{3*21} \rightarrow \frac{2*7**\sqrt{21}}{3*7*3} \rightarrow \frac{2\sqrt{21}}{9}$$

In all of the previous examples we just multiplied by "one in the form of" the denominator radical over the denominator radical.

$$\frac{3\sqrt{7}}{\sqrt{8}} \to \frac{3\sqrt{7}}{\sqrt{8}} * \frac{\sqrt{8}}{\sqrt{8}} \to \frac{3\sqrt{7*2*2*2}}{8} \to \frac{3*2\sqrt{7*2}}{8}$$

$$\to \frac{3*2\sqrt{14}}{2*4}$$

It is always easier to simplify (by factoring) ***BEFORE*** you multiply

$$\frac{3\sqrt{7}}{\sqrt{8}} \to \frac{3\sqrt{7}}{\sqrt{4}*\sqrt{2}} \to \frac{3\sqrt{7}}{2\sqrt{2}} * \frac{\sqrt{2}}{\sqrt{2}} \to \frac{3\sqrt{14}}{2*2} \to \frac{3\sqrt{14}}{4}$$

$$\frac{6\sqrt{5}}{3\sqrt{12}} \to \frac{3*2*\sqrt{5}}{3*\sqrt{4}*\sqrt{3}} \to \frac{2*\sqrt{5}}{2*\sqrt{3}} \to \frac{\sqrt{5}}{\sqrt{3}} * \frac{\sqrt{3}}{\sqrt{3}} \to \frac{\sqrt{15}}{3}$$

What about higher index numbers?

$$\frac{1}{\sqrt[3]{x}} * \frac{\sqrt[3]{x*x}}{\sqrt[3]{x*x}} \rightarrow \frac{\sqrt[3]{x*x}}{\sqrt[3]{x*x*x}} \rightarrow \frac{\sqrt[3]{x^2}}{x}$$

How many more 'x's are needed in the denominator radicand?

Remember: the cubed root of x-cubed equals x. $\sqrt[3]{x^3} = x$

We need two more x's under the denominator radical.

Using the multiply powers property we don't have to write out all the individual x's.

$$\frac{1}{\sqrt[3]{x}} * \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} \rightarrow \frac{\sqrt[3]{x^2}}{x}$$

Radicals CAN be written as Powers

Index

$$6\sqrt[5]{x}$$

Radicand

Coefficient

Exponent

$$6(x)^{1/5}$$

Base

Coefficient

Coefficient → Coefficient

Radicand → Base

Index → Denominator of the Exponent

The index number is the denominator of the exponent.

Are <u>radicals</u> related to <u>powers</u>?

$$3^{1/2} = \sqrt[2]{3}$$

$$5^{1/3} = \sqrt[3]{5}$$

$$3\sqrt[2]{y} = 3y^{1/2}$$

$$5\sqrt[3]{7} = 5(7)^{1/3}$$

$$\sqrt[2]{x} = x^{1/2}$$

$$\sqrt[3]{7} = 7^{1/3}$$

Multiplication (by a coefficient) is "repeated addition." This explains why coefficients of radicals become coefficients of powers.

$$\sqrt{y} = y^{1/2}$$

$$3\sqrt[2]{y} = \sqrt{y} + \sqrt{y} + \sqrt{y}$$

$$3y^{1/2} = y^{1/2} + y^{1/2} + y^{1/2}$$

None of these have coefficients!

What happens if there is a product under the radical?

$$\sqrt[2]{xy} = (xy)^{1/2}$$

$$5\sqrt[3]{3x} = 5(3x)^{1/3}$$

$$2\sqrt[4]{21mn} = 2(21mn)^{1/4}$$

How did we show that the index number applied to the entire product (radicand) when re-written in "power form"?

Power of a product → product inside parentheses with an exponent.

$$\sqrt[5]{x^2 y} = (x^2 y)^{1/5} = x^{2/5} y^{1/5}$$

$$6\sqrt[3]{3m^2} = 6(3m^2)^{1/3} = 6(3^{1/3})m^{2/3}$$

“Exponential Form” that has both a numerator and denominator

The exponent can be written as a rational number.

$$x^{5/2} = \sqrt[2]{x^5}$$

Numerator: Exponent of the base.

Denominator: Root of the base.

$$\sqrt[3]{2^2} = 2^{2/3}$$

Radical Form Exponential Form

Re-write in power form.

$\sqrt[2]{3m} \rightarrow (3m)^{1/2}$

$4\sqrt[3]{5y} \rightarrow 4(5y)^{1/3}$

$\sqrt[5]{x^3y^2} \rightarrow (x^3y^2)^{1/5} \rightarrow x^{3/5}y^{2/5}$

$5\sqrt[4]{3m^2} \rightarrow 5(3m^2)^{\frac{1}{4}}$

Rewrite in "radical form"

$m^{1/5} \rightarrow \sqrt[5]{m}$

$3nm^{1/4} \rightarrow 3n\sqrt[4]{m}$

$2(18n^2)^{1/6} \rightarrow 2\sqrt[6]{18n^2}$

$5(4x^2y^6)^{1/3} \rightarrow 5(4x^2y^6)^{\frac{1}{4}}$

Multiply Powers Property

$$y^2 * y^3 = ? \quad = y^{2+3} \quad = y^5$$

When multiplying "same based powers" add the exponents.

$$x^{\frac{2}{3}} * x^{\frac{3}{4}} \quad \rightarrow x^{\frac{2}{3}+\frac{3}{4}} \quad \rightarrow x^{\frac{17}{12}}$$

Yes, you must be able to add fractions

Exponent of a Power Property

$$\left(y^2\right)^3 = ? \quad = y^{2*3} \quad = y^6$$

When multiplying "same based powers" add the exponents.

$$\left(y^{1/2}\right)^{2/3} = y^{\frac{1}{2}*\frac{2}{3}} = y^{\frac{1}{3}}$$

Rational Exponents in the Denominator

$$\frac{1}{y^{1/2}} = \frac{1}{\sqrt{y}}$$

Rational exponent in the denominator means irrational denominator, which we rationalize

$$\frac{1}{y^{1/2}} = \frac{1}{\sqrt{y}} * \frac{\sqrt{y}}{\sqrt{y}} = \frac{\sqrt{y}}{y}$$

Rational exponent in the denominator → what is the next bigger natural number from ½ ?

1

$$\frac{1}{y^{1/2}} * \frac{y^{1/2}}{y^{1/2}} = \frac{y^{1/2}}{y}$$

What number do you add to ½ to get 1?

In order to add a number to an exponent, you have to multiply by a same-based power with the exponent you are trying to add.

Negative Exponent Property

Grab and drag same-based powers to be next to each other.

$$\frac{x^2 y^{2/3}}{y^{-1/2}} = x^2 y^{2/3} y^{1/2} = x^2 y^{\frac{2}{3}+\frac{1}{2}} = x^2 y^{\frac{4}{6}+\frac{3}{6}} = x^2 y^{\frac{7}{6}}$$

$$\frac{2x^{\frac{1}{3}}}{x^{\frac{2}{3}}} \rightarrow \frac{2}{x^{\frac{2}{3}} x^{-\frac{1}{3}}} \rightarrow \frac{2}{x^{\frac{1}{3}}}$$

Not allowed to have rational exponents in the denominator

$$\rightarrow 2x^{1/3} x^{-2/3} \quad \rightarrow 2x^{-1/3}$$

Not allowed to have negative exponents.

Negative Exponent Property

$$\frac{2x^{\frac{1}{3}}}{x^{\frac{2}{3}}} \rightarrow \frac{2}{x^{\frac{2}{3}}x^{-\frac{1}{3}}} \rightarrow \frac{2}{x^{\frac{1}{3}}}$$

What is the next bigger whole number than 1/3 ? 1

What number do you add to 1/3 to get 1? 2/3

$$\rightarrow \frac{2}{x^{\frac{1}{3}}} * \frac{x^{\frac{2}{3}}}{x^{\frac{2}{3}}} \rightarrow \frac{2x^{\frac{2}{3}}}{x^{\frac{1}{3}+\frac{2}{3}}}$$

$$\rightarrow \frac{2x^{\frac{2}{3}}}{x}$$

Multiply by one "in the form of" a same-base power whose exponent is 2/3 (both numerator and denominator)