Math-3
Lesson 2-5
Analyzing Polynomial Equations

Polynomial: An equation (or an expression) with same-base powers being added that are raised to a *natural number* exponent.

Example: $y = 8x^5 + 5x^4 + 9x^3 + x^2 + 2x + 3$

Not a polynomial $y = x^{0.5} + 3x^{2/3} + 6\sqrt{x}$

Lead coefficient: the coefficient of the largest power.

$$y = -8x^5 + 5x^4 + 9x^3 + x^2 + 2x + 3$$

Degree: the largest exponent of the polynomial.

Standard Form Polynomial A polynomial ordered so that the exponents get smaller from the left-most term to the right-most term.

$$y = \underline{8x^5} + \underline{5x^4} + \underline{9x^3} + \underline{x^2} + \underline{2x} + \underline{3}$$

Term: powers (or the constant) separated by either a '+' or '-' symbol.

Number of terms: If all terms are present, a 2nd degree polynomial as 3 terms in standard form.

$$y = 2x^2 - 4x + 5$$

If you include the number *zero* as a possible coefficient, an "n-th degree polynomial has n+1 terms (i.e., a 3rd degree has 4 terms).

$$y = 4x^3 + 0x^2 - 4x + 5$$

Intercept Form Polynomial A polynomial that has been factored into *linear factors*, from which you can identify the input values that make the output value equal to zero.

Example: $$y = 6(x+4)(x+3)(x-2i)(x+2i)$$

Linear factors: the exponent of the power is a '1'.

$y = mx + b$ Is a linear equation so (x + 2) is a linear factor

Solve by factoring: If the equation has only one variable ('y' has already been set to zero), solve by factoring means to convert a standard form polynomial into intercept form (by factoring) and then identifying the zeroes of the polynomial.

$$y = 6x^4 + 42x^3 + 96x^2 + 28x + 48$$

Zeroes of Polynomials: come from linear factors.

$$0 = 6(x+4)(x+3)(x-2i)(x+2i)$$

$$x = -4 \quad x = -3 \quad x = 2i \quad x = -2i$$

If the polynomial is already in intercept form: just find the zeroes.

$$0 = (x+5)(x-2)(x-\sqrt{3})(x+\sqrt{3})$$

$$x = -5 \quad x = 2 \quad x = \sqrt{3} \quad x = -\sqrt{3}$$

Given the zeroes, write the equation of the Polynomial:

$x = -2 \qquad x = 3 \qquad x = 6$

$y = (x+2)(x-3)(x-6)$

Polynomial: An equation (or an expression) with same-base powers being added that are raised to a *non-negative integer exponents.* → Convert to "standard form."

$y = a(x+2)(x-3)(x-6)$ → Assume VSF = 1

$y = (x+2)(x^2 - 9x + 18)$

	x^2	$-9x$	18
x	x^3	$-9x^2$	$18x$
2	$2x^2$	$-18x$	36

$y = x^3 - 7x^2 + 36$

Fundamental Theorem of Algebra: If a polynomial has a degree of "n", then the polynomial has "n" zeroes (provided that repeat zeroes, called "multiplicities" are counted separately).

$$y = 6x^4 + 42x^3 + 96x^2 + 28x + 48$$

"4th Degree" → 4 zeroes x = -4, -3, 2i, -2i

Multiplicity: the number of times a zero is repeated for a polynomial.

$$y = (x-1)^2 = (x-1)(x-1)$$

x = 1 is a "zero" of the polynomial twice.

x= 1 is a zero of the polynomial with multiplicity 2

Linear Factorization Theorem: If a polynomial has a degree of "n", then the polynomial can be factored into "n" linear factors.

$$y = 6(x+4)(x+3)(x-2i)(x+2i)$$

Since each linear factor has one zero, these two theorems are almost saying the same thing.

Quadratic Formula: gives zeroes of 2nd degree polynomials.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Complex Conjugates Theorem: If (a + bi) is a zero of a polynomial, then its complex conjugate (a - bi) is also a zero of f(x).

Example: $y = x^2 - 4x + 5$

Factorization: $y = (x - 2 - 3i)(x - 2 + 3i)$

Zeroes: $x = 2 - 3i, \quad x = 2 + 3i$

Example: $y = x^2 + 4$

Factorization: $y = (x + 2i)(x - 2i)$

Zeroes: $x = 2i, -2i$

Quadratic Formula: gives zeroes of 2nd degree polynomials.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Irrational Roots (Zeroes) Theorem: If an irrational number is the zero of a polynomial, then the conjugate of the irrational number is also a zero.

Example: $y = x^2 + 4x + 1$

Factorization: $y = (x + 2 - \sqrt{3})(x + 2 + \sqrt{3})$

Zeroes: $x = 2 - \sqrt{3}$, $x = 2 + \sqrt{3}$

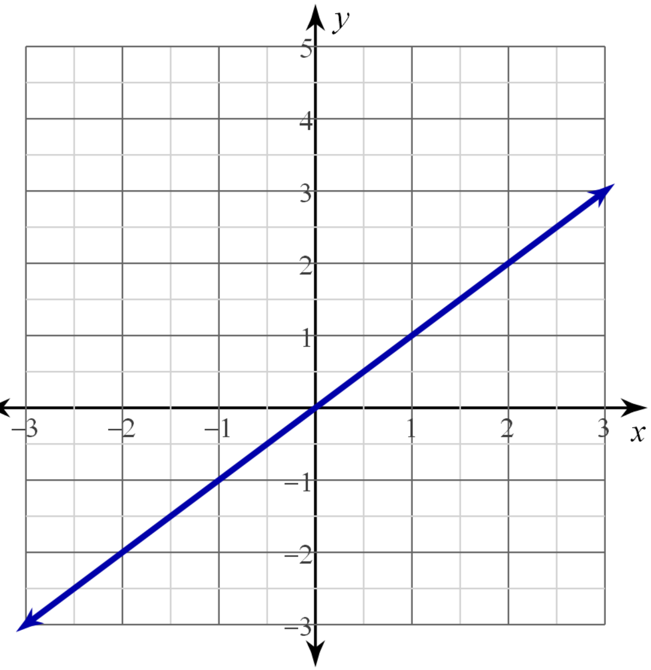
Example: $y = x^2 + 4$

Factorization: $y = (x + 2i)(x - 2i)$

Zeroes: $3 - \sqrt{2}$, $3 + \sqrt{2}$

Analyzing Polynomials

What is the "end behavior" of the graph?



$$y = x$$

(right end goes up)

$$\text{as } x \to +\infty, \; y \to ?$$

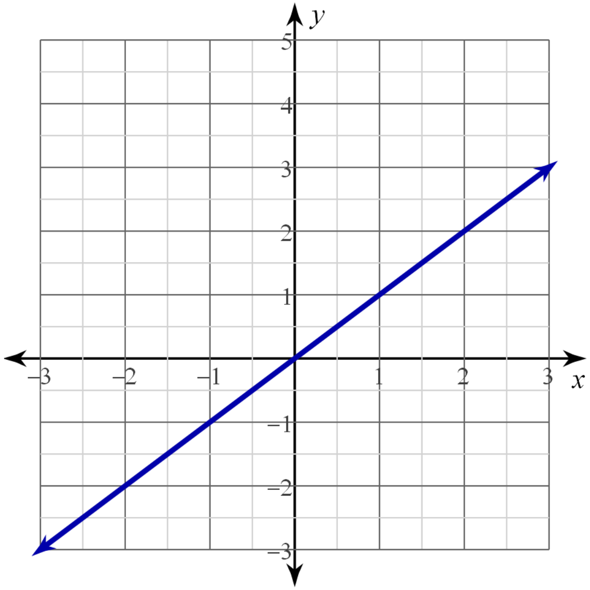
y → ∞

(left end goes down)

$$\text{as } x \to -\infty, \; y \to ?$$

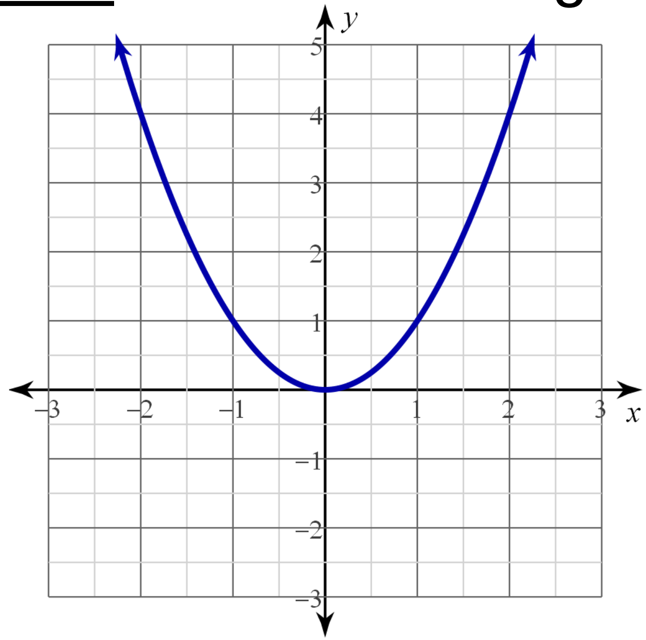
y → -∞

Do you notice a <u>pattern</u> between the degree and the end behavior?



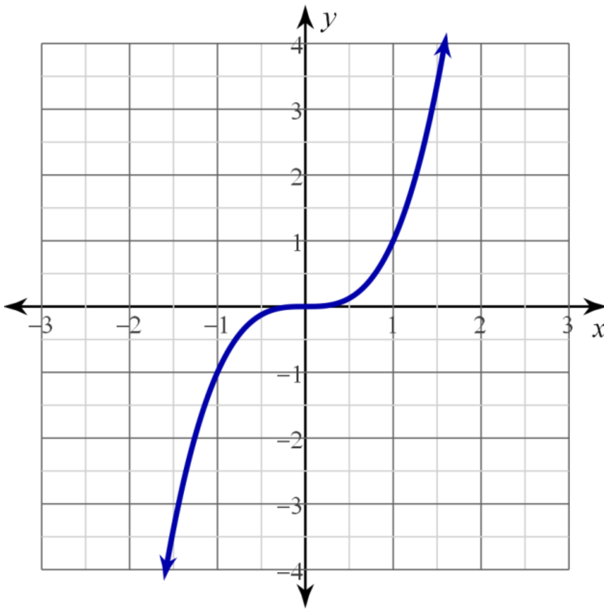
$$y = x$$

(up right, down left)



$$y = x^2$$

(up right, up left)



$$y = x^3$$

(up right, down left)

<u>Odd degree</u>: (up right, down left)

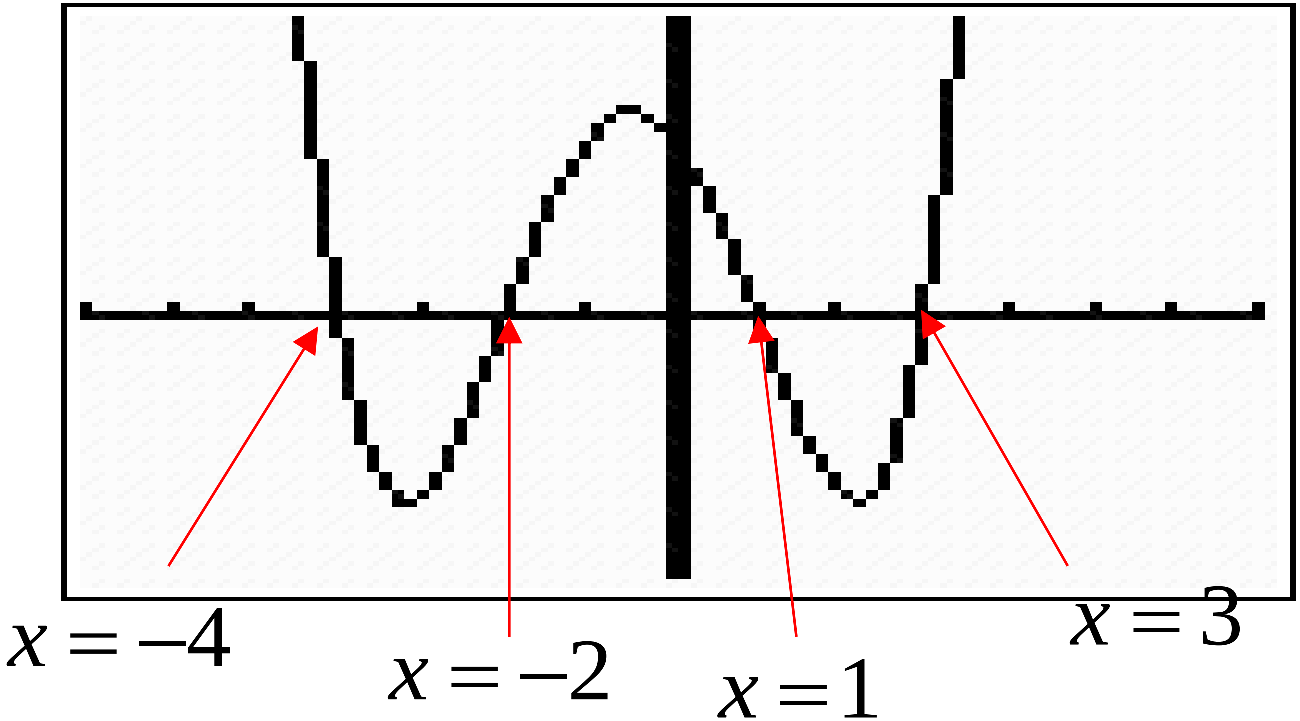
<u>even degree</u>: (up right, up left)

4^{th} Degree Polynomial (even degree)

$$y = (2x + 8)(3x + 6)(x - 1)(x - 3)$$

"left * left * left * left" = ? $= 6x^4$

$$y = 6x^4 + \ldots(\text{other terms})$$



Degree vs. End Behavior

$$y = x^5 + x^4 + x^3 + x^2 + x + 1$$

Pick a very large input value: 1,000,000 = 10^6 then compare each term.

$$(10^6)^5 = 10^{30}$$

$$(10^6)^4 = 10^{24}$$

$$(10^6)^3 = 10^{18}$$

$$(10^6)^2 = 10^{12}$$

$$(10^6 = 10^6$$

$$1 = 10^0$$

Compare the largest two powers.

$$10^{30} = 10^{24} * 10^6$$

Input = 1,000,000 → output of the 1st term is 1,000,000 times as big as the output of the 2nd term.

$$y = x^5 + x^4 + x^3 + x^2 + x + 1$$

As x → ∞, x^5 becomes the dominant term

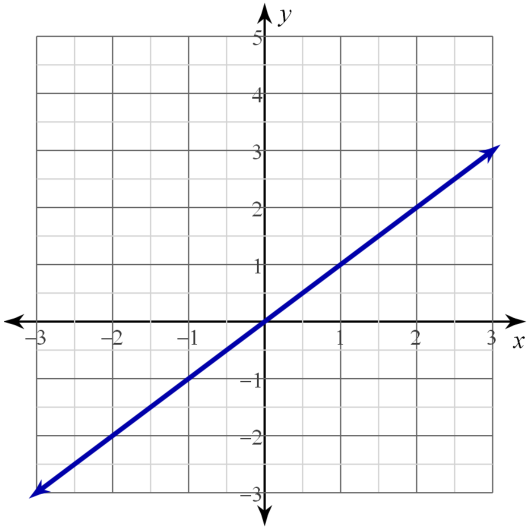
→ all the other terms become insignificant compared to x^5.

How does this affect end behavior?

The lead term determines end behavior!!!

What is the end-behavior of an odd-degree polynomial?

(right end up, left end down)



Multiply Powers Property of Exponents. $x^2 * x^3 = x^5$

<u>DO NOT</u> think of exponents as a power! The base AND its exponent combined is a power.

$$y = 3(x-2)^3(x+4)^2(x-\sqrt{5})(x+\sqrt{5})(x-3i)(x+3i)$$

What are the zeroes and their multiplicities?

Zeroes: x = 2 (multiplicity 3), -4 (mult. 2), etc.

What is the degree of this polynomial?

(3rd degree) *(2nd degree)*(1st)(1st)(1st)(1st) = 9th degree

→ Odd degree

Polynomials in intercept form

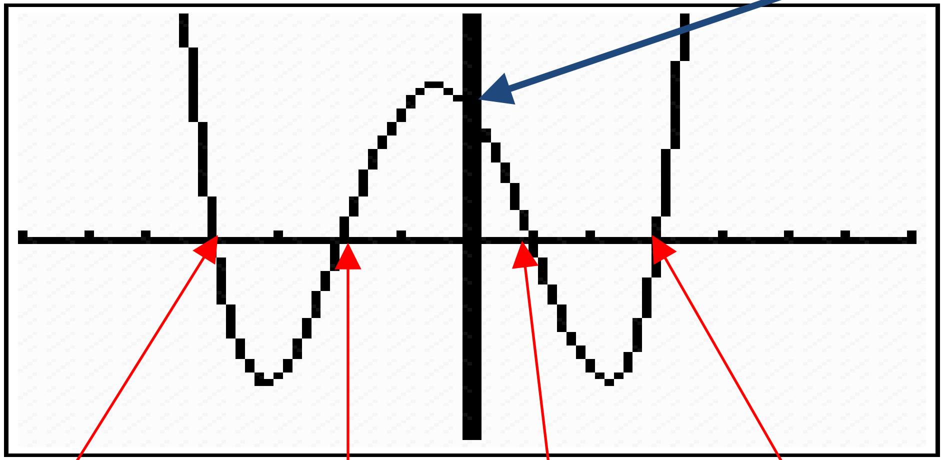
$$y=(x-1)(2x+2)(x-3)(2x+8)$$

$$y=12x^4+\ldots(\text{other terms})\ldots+48$$

“right * right * right * right” = ?

$$=48$$

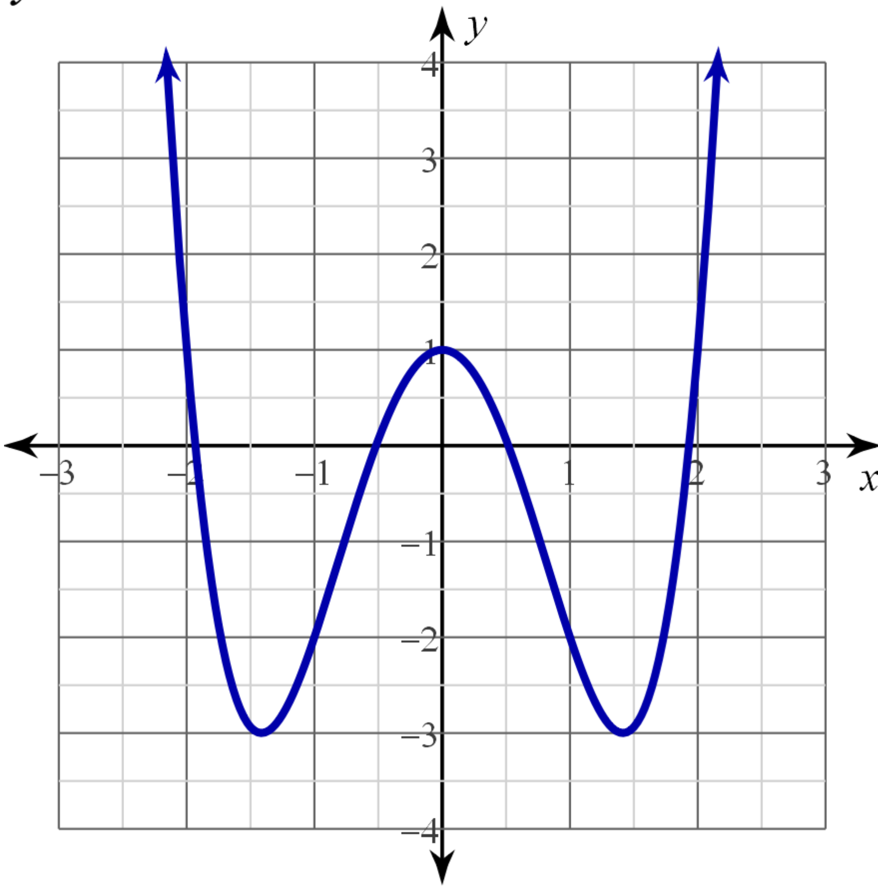
y-intercept = 48



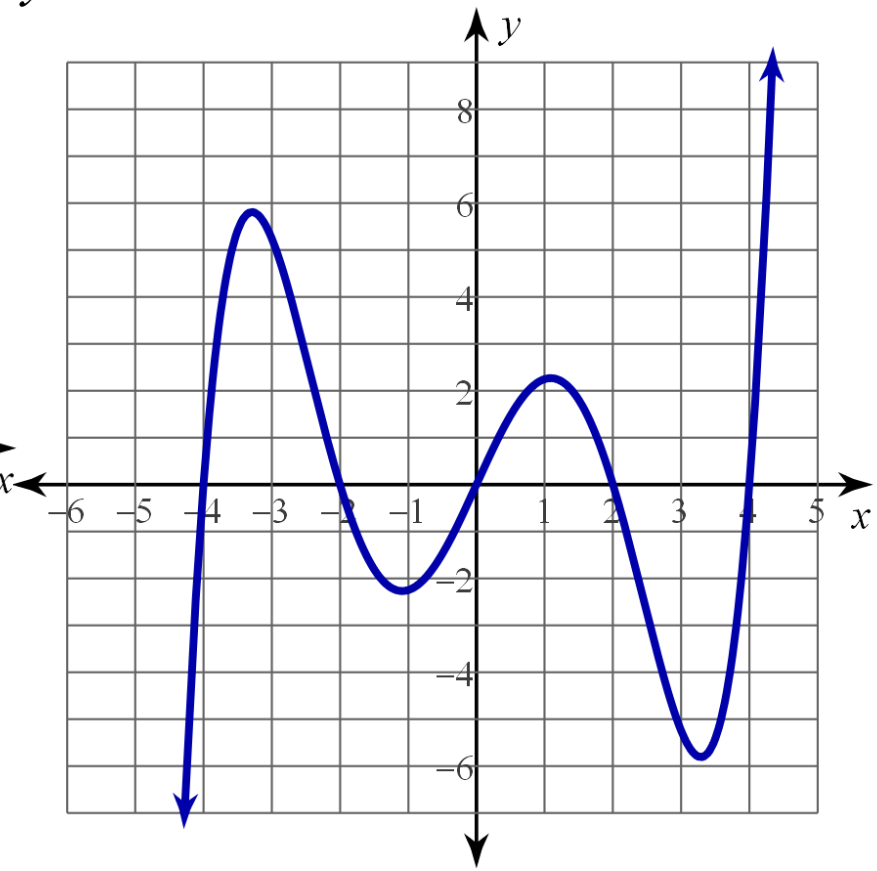
4th degree polynomial → “4 zeroes”

Max number of x-intercepts?

$$y = x^4 - 4x^2 + 1$$



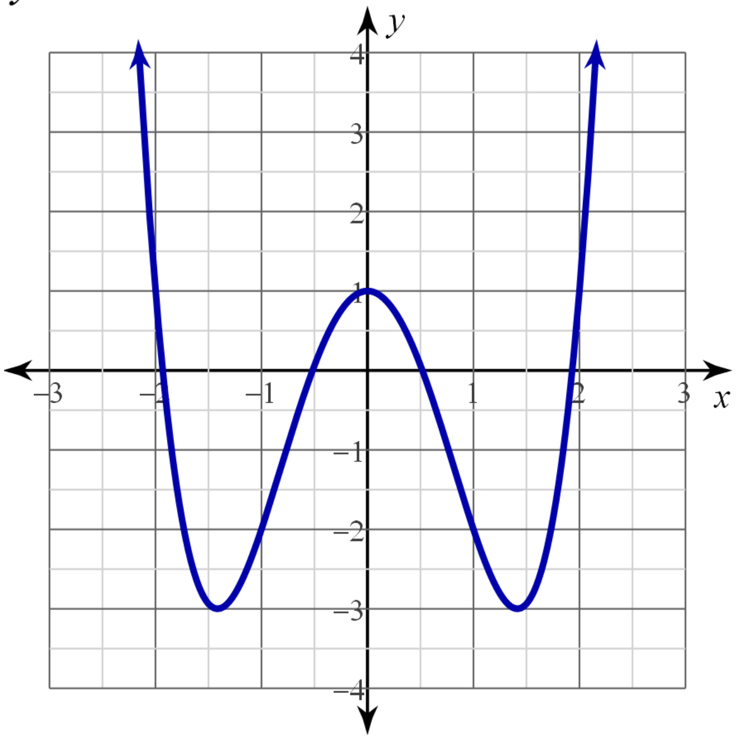
$$y = x^5 - x^4 - 4x^2 + 1$$



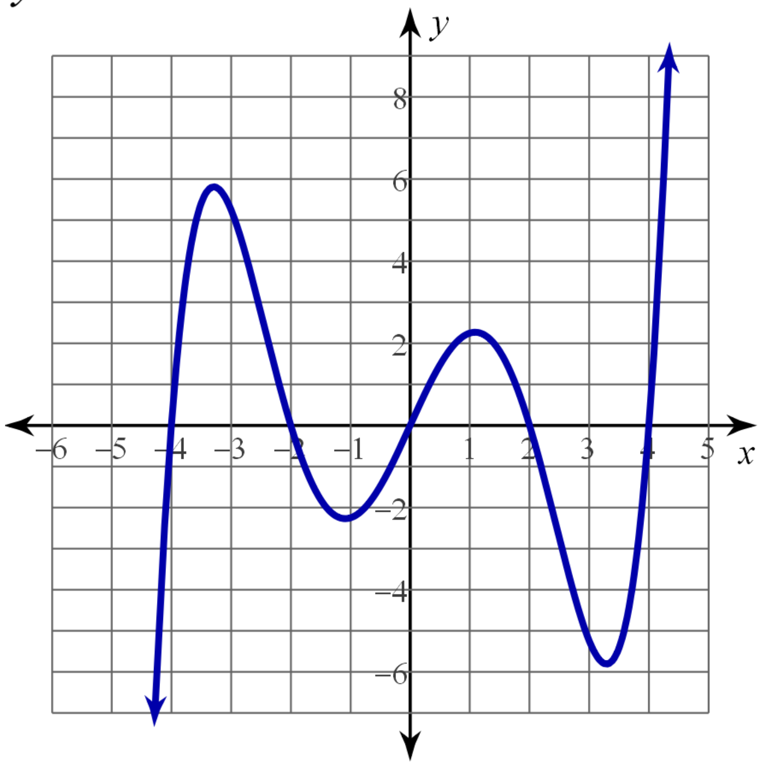
The <u>degree</u> of the polynomial equals the <u>number of zeroes</u> AND gives you the <u>max number of x-intercepts</u> (real number zeroes).

Polynomial Degree → End Behavior?

$y = x^4 - 4x^2 + 1$



$y = x^5 - x^4 - 4x^2 + 1$

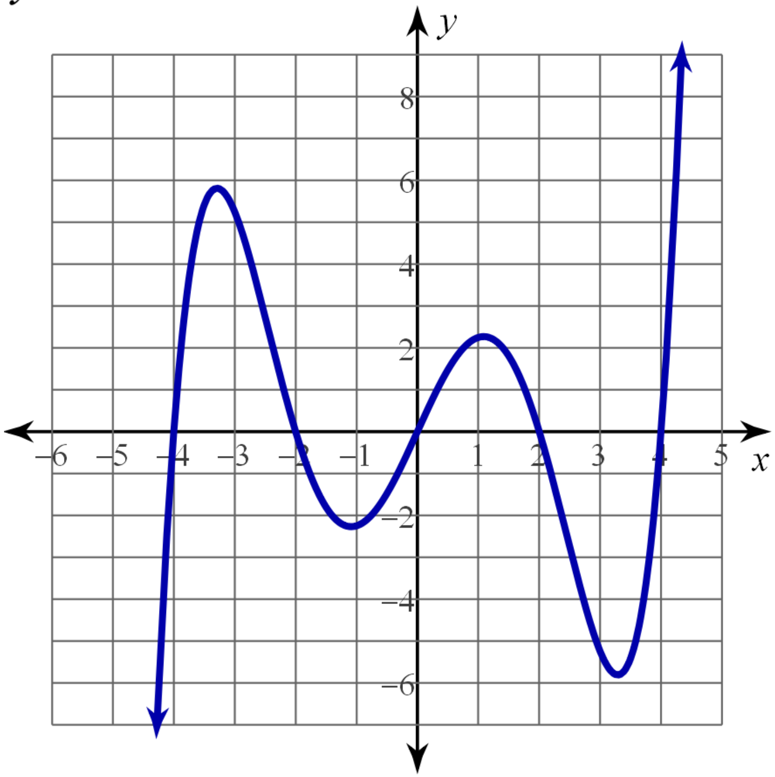


All <u>even degree</u> polynomials have the <u>same end behavior!</u>

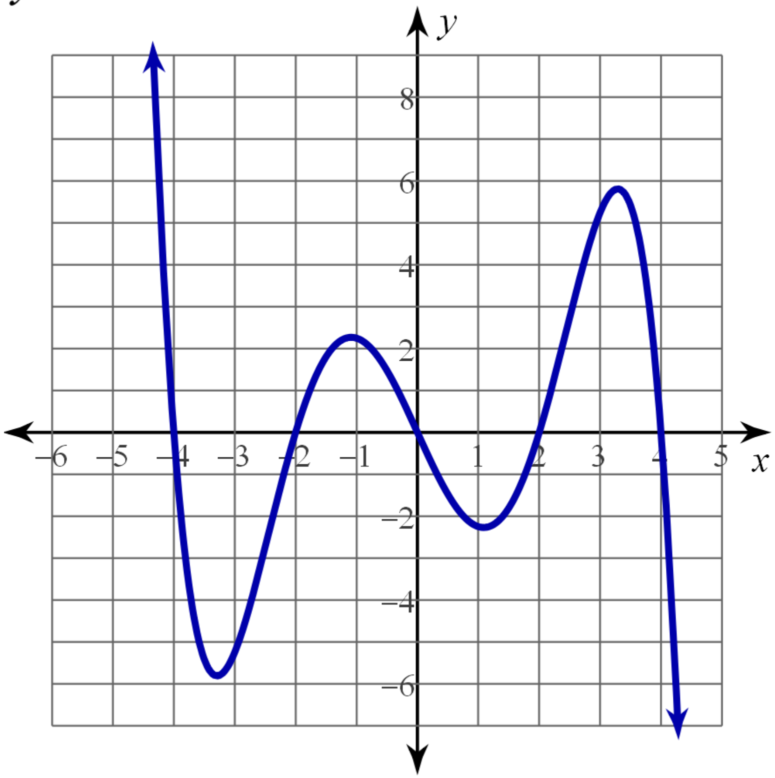
All <u>odd degree</u> polynomials have the <u>same end behavior!</u>

Lead Coefficient & Degree→ End Behavior?

$y = x^5 - x^4 - 4x^2 + 1$



$y = -x^5 - x^4 - 4x^2 + 1$

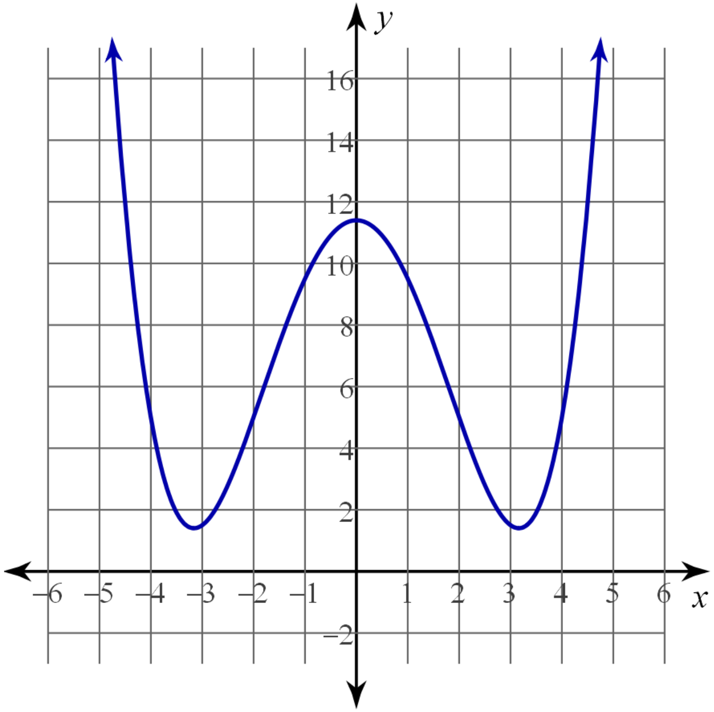


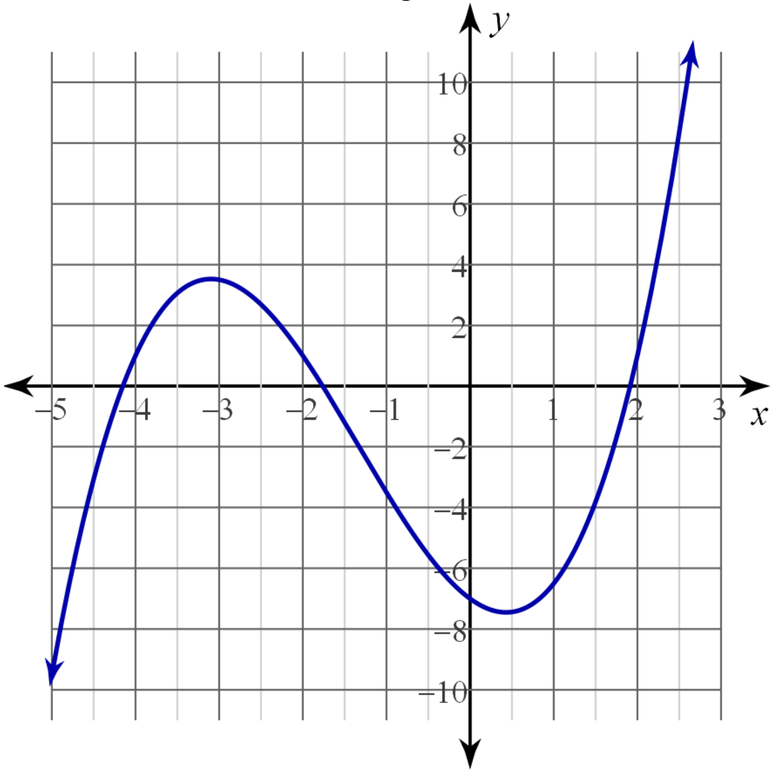
All <u>odd degree</u> polynomials have the <u>same end behavior!</u>

<u>negative lead coefficient:</u> reflection across the x-axis, all negative-odd polynomials have the <u>same end behavior!</u>

How many real zeroes will the polynomial have?

Does an even degree polynomial necessarily cross the x-axis?





Does an odd degree polynomial necessarily cross the x-axis?

Describe the end behavior (up left/right, up left down right, etc.)

$$f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8$$

Positive lead coefficient, even degree ↑ left ↑ right

$$f(x) = -7x^3 - 8x^2 + 14x + 8$$

negative lead coefficient, odd degree ↑ left ↓ right

Make a table of the possible zeroes by category

Degree	Real zeroes	Imaginary Zeroes
2	2	0
	1 (mult 2)	0
	1	1 Not possible
	0	2

Not possible because imaginary zeroes always come conjugate pairs (Complex Conjugates Theorem)

If we count multiplicities as separate solutions it is easier.

Degree	Real zeroes	Imaginary Zeroes
2	2	0
	1	1 Not possible
	0	2

Degree	Real zeroes	Imaginary Zeroes
3	0	3 Not possible
	1	2
	2	1 Not possible
	3	0

Degree	Real zeroes	Imaginary Zeroes
4	0	4
	1	3 Not possible
	2	2
	3	1 Not possible
	4	0

The General Shape of the Graph of a Polynomial

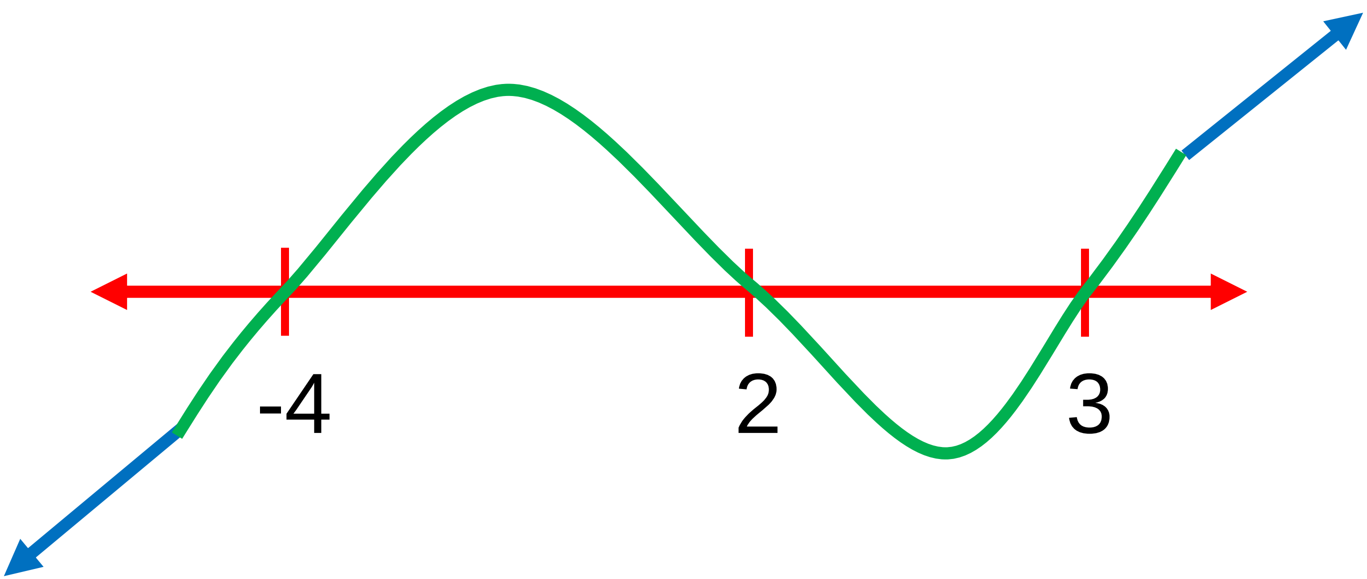
$$f(x) = (x-2)(x-3)(x+4)$$

zeroes: x = 2, 3, and -4.

All are real numbers. → All are x-intercepts.

positive lead coefficient and an odd degree.

The end behavior is ↓ left, ↑ right?



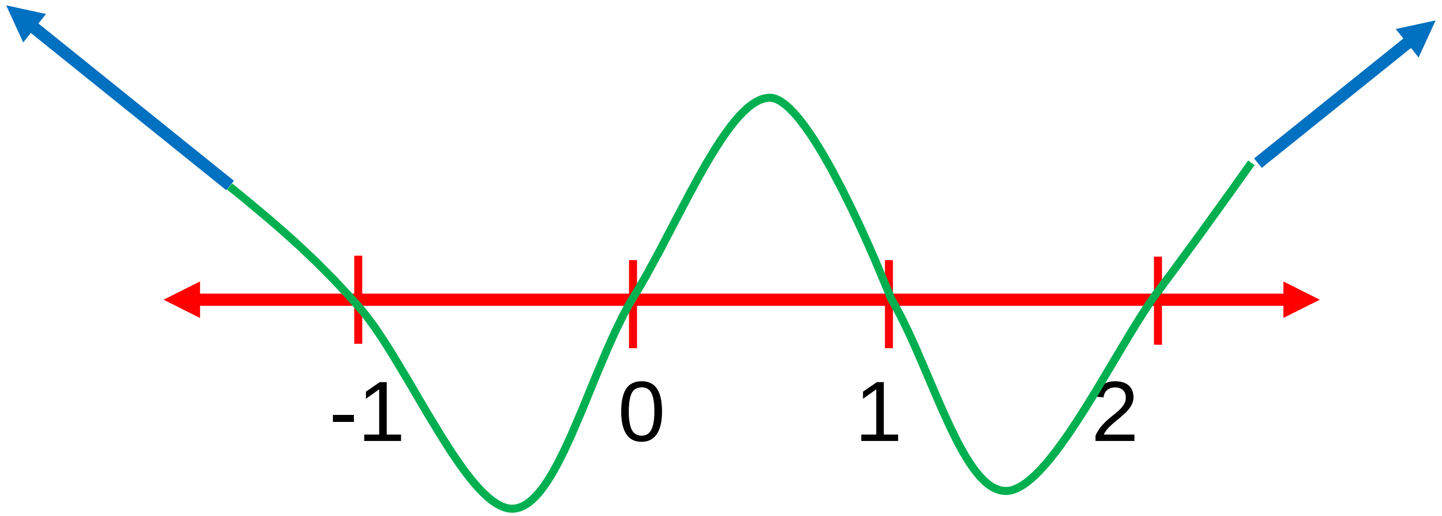
$$f(x) = x(x+1)(x-1)(x-2)$$

zeroes: x = 0, -1, 1, and 2.

All are real numbers. All are x-intercepts.

positive lead coefficient and an even degree.

The end behavior is up right, up left?



$$f(x) = (x+1)^2(x+3)(x-4)$$

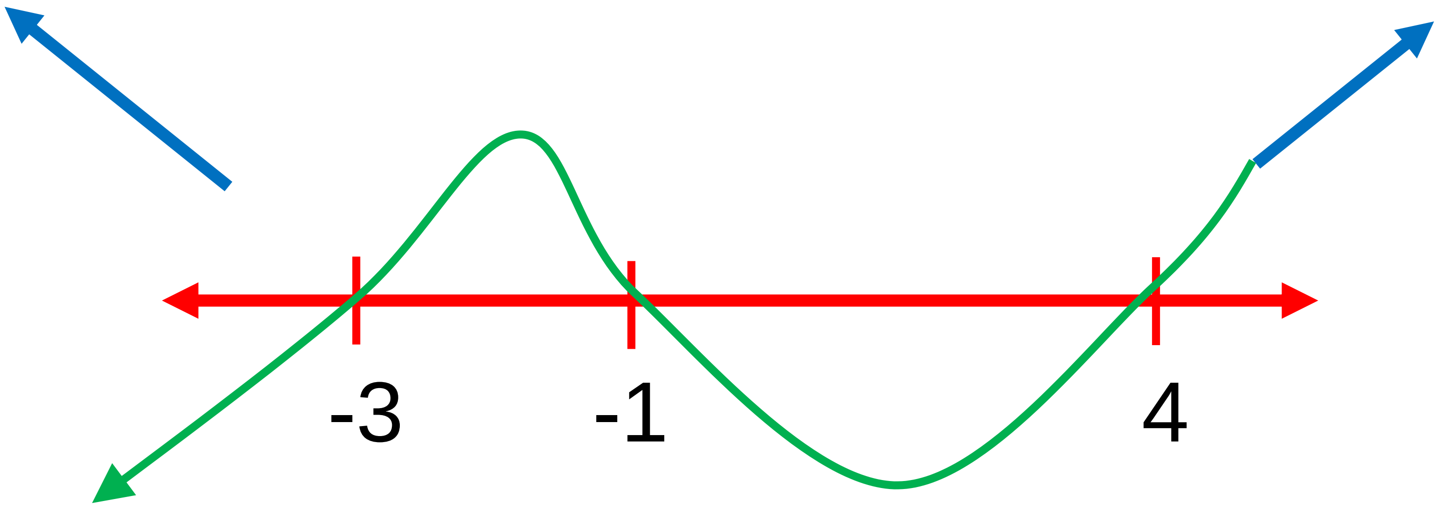
zeroes: x = -1, -1, -3, and 4.

All are <u>real numbers.</u>

All are <u>x-intercepts.</u>

<u>positive lead coefficient</u> and an <u>even degree.</u>

The <u>end behavior</u> is **Up right, up left**?



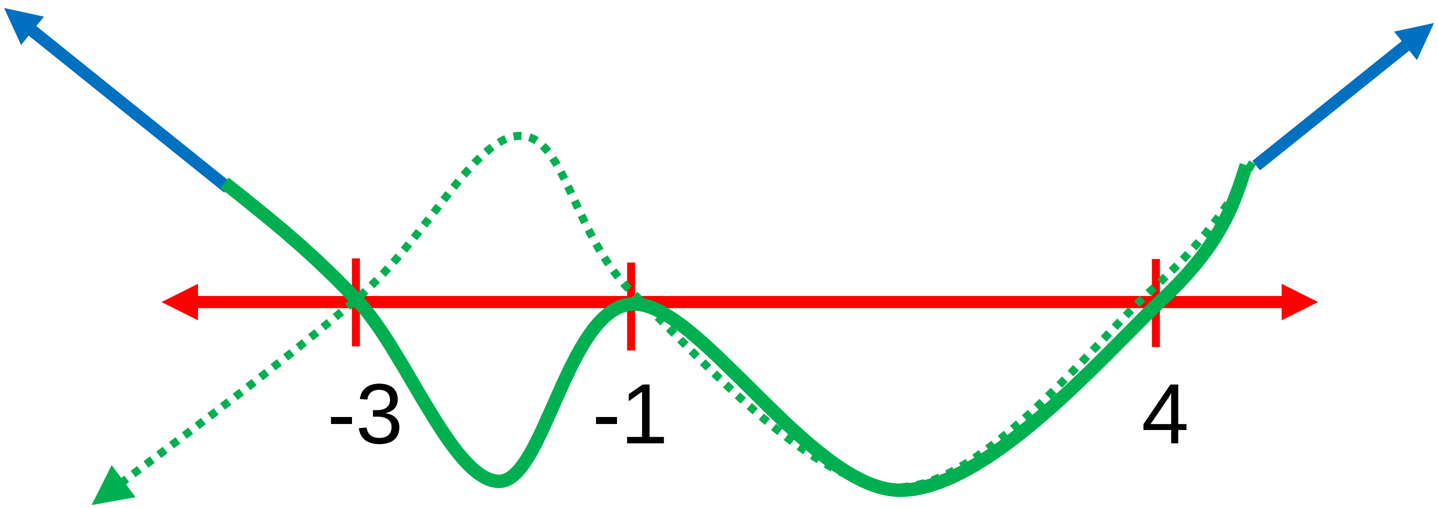
Why doesn't the "end behavior" line up?

$$f(x) = (x+1)^2(x+3)(x-4)$$

It has the following zeroes: x = -1, -1, -3, and 4.

The zero with an EVEN "multiplicity will just "kiss" the x-axis.

Remember $y = x^2$?



$$f(x) = (x+2i)(x-2i)(x-4)^2(x+2)$$

It has the following zeroes: x = 2i, -2i, 4, 4, and -2.

Only 4 and -2 are real numbers. These are x-intercepts.

positive lead coefficient and an odd degree.

The end behavior is ____Up right, down left________?

The graph "kisses" at x = 4