Math-3A
Lesson 3-1
Analyzing Polynomial Equations

Polynomial: An equation (or an expression) with same-base powers being added that are raised to a *natural number exponent.*

Example: $y = 8x^5 + 5x^4 + 9x^3 + x^2 + 2x + 3$

Not a polynomial $y = x^{0.5} + 3x^{2/3} + 6\sqrt{x}$

Lead coefficient: the coefficient of the largest power.

$$y = -8x^5 + 5x^4 + 9x^3 + x^2 + 2x + 3$$

Degree: the largest exponent of the polynomial.

Standard Form Polynomial A polynomial ordered so that the exponents get smaller from the left-most term to the right-most term.

$$y = \underline{8x^5} + \underline{5x^4} + \underline{9x^3} + \underline{x^2} + \underline{2x} + \underline{3}$$

Term: powers (or the constant) separated by either a '+' or '-' symbol.

Number of terms: If all terms are present, a 2nd degree polynomial as 3 terms in standard form.

$$y = 2x^2 - 4x + 5$$

If you include the number *zero* as a possible coefficient, an "n-th degree polynomial has n+1 terms (i.e., a 3rd degree has 4 terms).

$$y = 4x^3 + 0x^2 - 4x + 5$$

Intercept Form Polynomial A polynomial that has been factored into *linear factors*, from which you can identify the input values that make the output value equal to zero.

Example: $$y = 6(x+4)(x+3)(x-2i)(x+2i)$$

Linear factors: the exponent of the power is a '1'.

Why do we call these linear factors?

$y = mx + b$ Is a linear equation so (x + 2) is a linear factor

Fundamental Theorem of Algebra: If a polynomial has a degree of "n", then the polynomial has "n" zeroes (provided that repeat zeroes, called "multiplicities" are counted separately).

$$y = 6x^4 + 42x^3 + 96x^2 + 28x + 48$$

"4th Degree" → 4 zeroes   x = -4, -3, 2i, -2i

Linear Factorization Theorem: If a polynomial has a degree of "n", then the polynomial can be factored into "n" linear factors.

$$y = 6(x+4)(x+3)(x-2i)(x+2i)$$

Since each linear factor has one zero, these two theorems are almost saying the same thing.

How can you tell if there are zeroes that are multiplicities?

$$y = 3(x-2)^3(x+4)^2(x-\sqrt{5})(x+\sqrt{5})(x-3i)(x+3i)$$

Zeroes:

$$(x-2)^3 \rightarrow (x-2)(x-2)(x-2)$$

x = 2 (multiplicity 3)

$$(x+4)^2 \rightarrow (x+4)(x+4)$$

x = 4 (multiplicity 2)

$$(x-\sqrt{5})(x+\sqrt{5})(x-3i)(x+3i)$$

4 single multiplicity zeroes

What is the degree of this polynomial?

(3rd degree) *(2nd)*(1st)(1st)(1st)(1st) = 9th degree

Product of powers property of exponents!

Solve by factoring: If the equation has only one variable ('y' has already been set to zero), solve by factoring means to convert a standard form polynomial into intercept form (by factoring) and then identifying the zeroes of the polynomial.

$$0 = 6(x+4)(x+3)(x-2i)(x+2i)$$

$$x = -4 \quad x = -3 \quad x = 2i \quad x = -2i$$

Find the zeroes: means that the equation has two variables.
1st step: set y = 0, then solve by factoring.

$$y = 6x^4 + 42x^3 + 96x^2 + 28x + 48$$

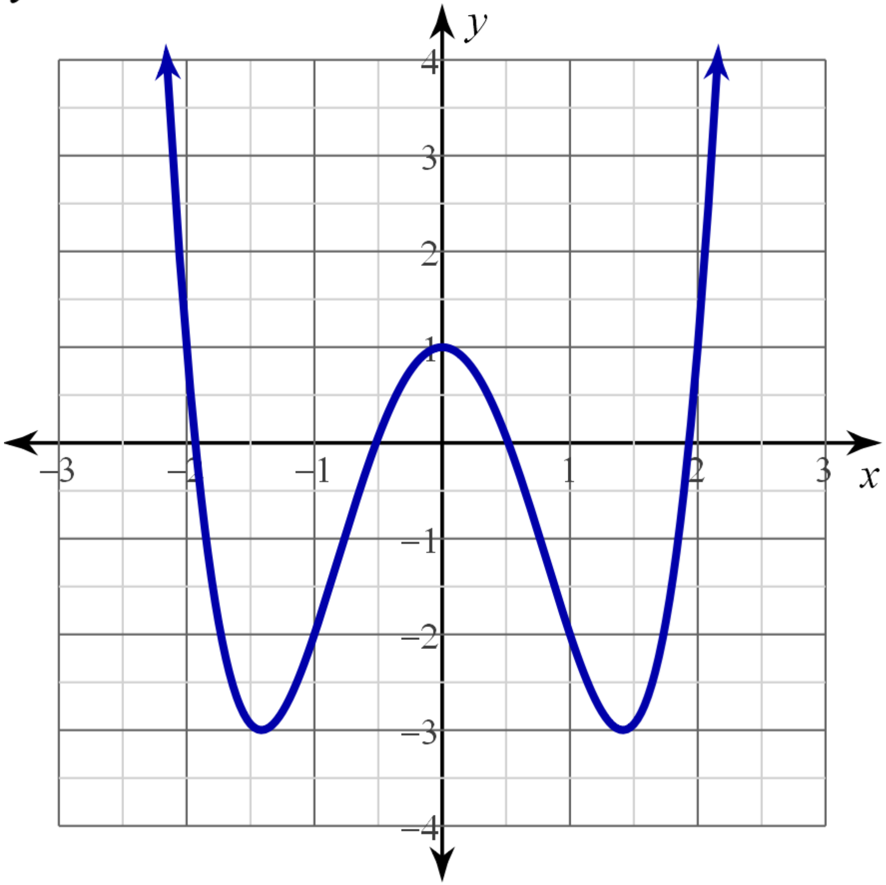
If the polynomial is already in intercept form: "solve by factoring " means just find the zeroes.

$$0 = (x+5)(x-2)(x-\sqrt{3})(x+\sqrt{3})$$

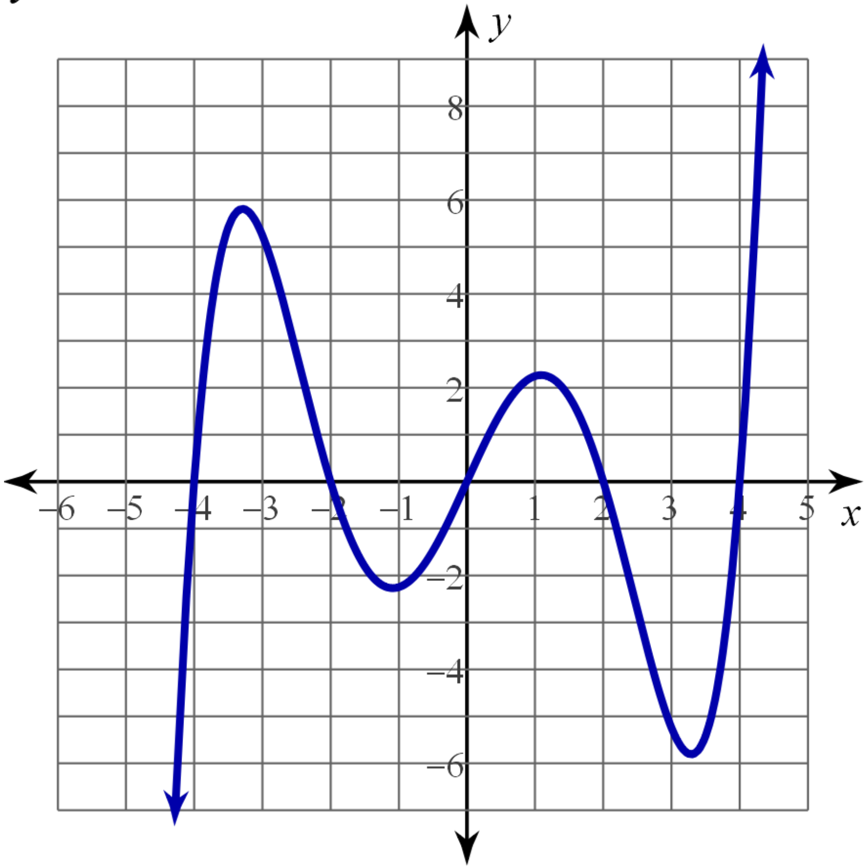
$$x = -5 \quad x = 2 \quad x = \sqrt{3} \quad x = -\sqrt{3}$$

Max number of x-intercepts?

$$y = x^4 - 4x^2 + 1$$

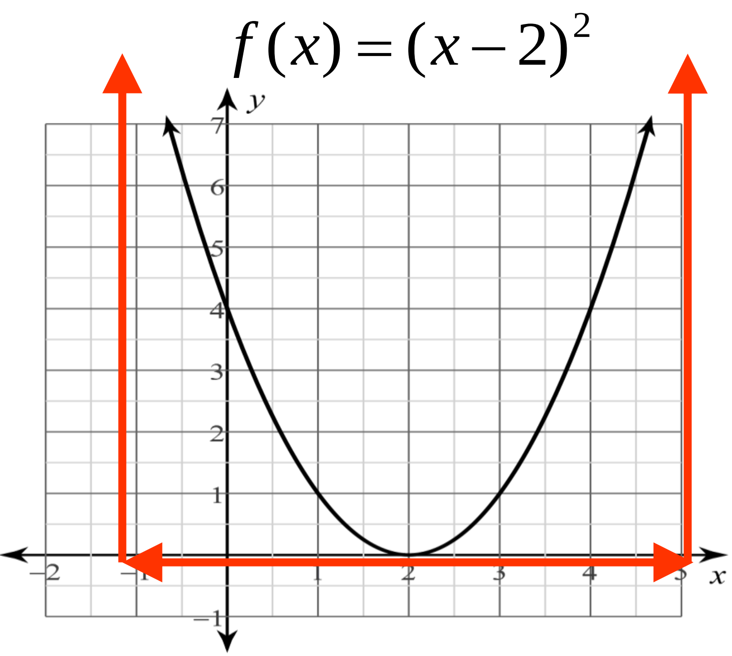


$$y = x^5 - x^4 - 4x^2 + 1$$



The <u>degree</u> of the polynomial equals the <u>number of zeroes</u> AND gives you the <u>max number of x-intercepts</u> (real number zeroes).

“End Behavior”



The “end behavior” of a function means:

“on the right end of the graph is the y-value going UP or DOWN?
And
“on the left end of the graph,
is the y-value going UP or DOWN?

In English we could say: “up on right, up on left”

As ‘x’ gets bigger (right end) ‘y’ gets bigger (goes upward)

As ‘x’ gets smaller (left end), ‘y’ gets bigger (goes upward) “end behavior”

$$\text{as } x \to +\infty, \; y \to ? \qquad \text{as } x \to -\infty, \; y \to ?$$

Which of the following transformations affect end behavior? If so, how?

Left or right shift? no

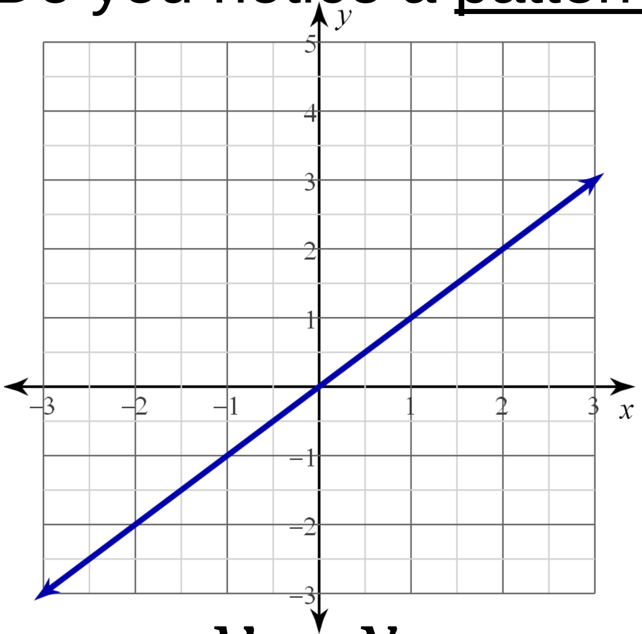
Up or down shift? no

Vertical stretching? no

Reflection across x-axis? yes

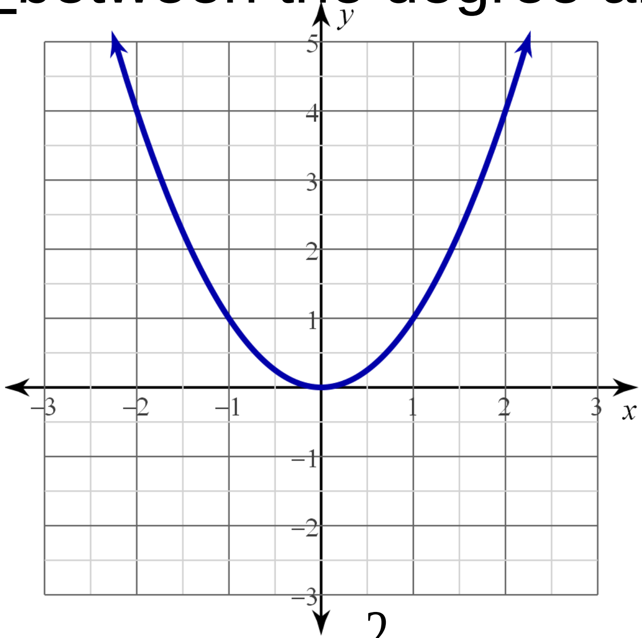
→ Opposite end-behavior as the parent function.

Do you notice a pattern between the degree and the end behavior?



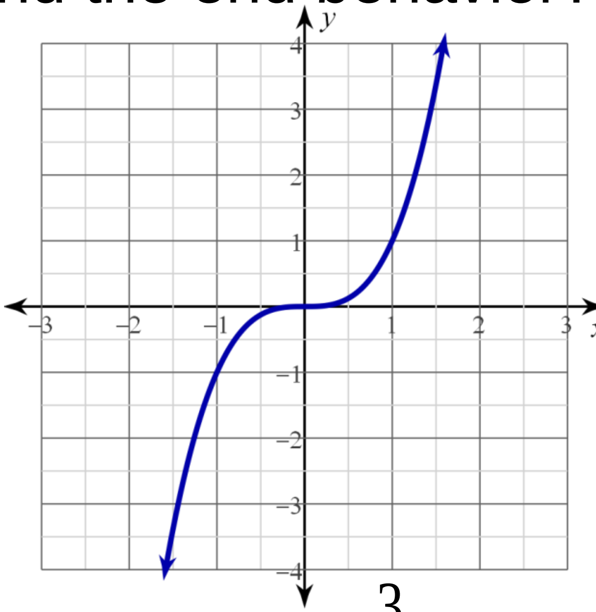
$y = x$

(up right, down left)



$y = x^2$

(up right, up left)



$y = x^3$

(up right, down left)

Odd degree: (up right, down left)

even degree: (up right, up left)

How would end-behavior change if the function has been reflected across the x-axis?

Degree vs. End Behavior

$$y = x^5 + x^4 + x^3 + x^2 + x + 1$$

Pick a very large input value: 1,000,000 = 10^6 then compare each term.

$$(10^6)^5 = 10^{30}$$

$$(10^6)^4 = 10^{24}$$

$$(10^6)^3 = 10^{18}$$

$$(10^6)^2 = 10^{12}$$

$$(10^6 = 10^6$$

$$1 = 10^0$$

Compare the largest two powers.

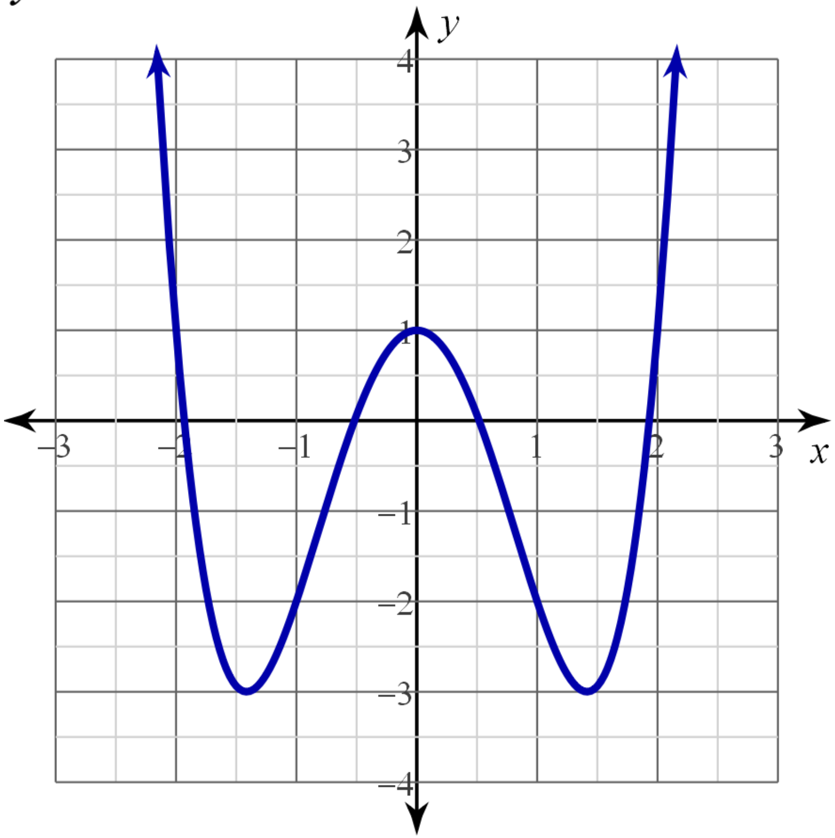
$$10^{30} = 10^{24} * 10^6$$

For x = 1,000,000 → x^6 is 1,000,000 times larger than x^5

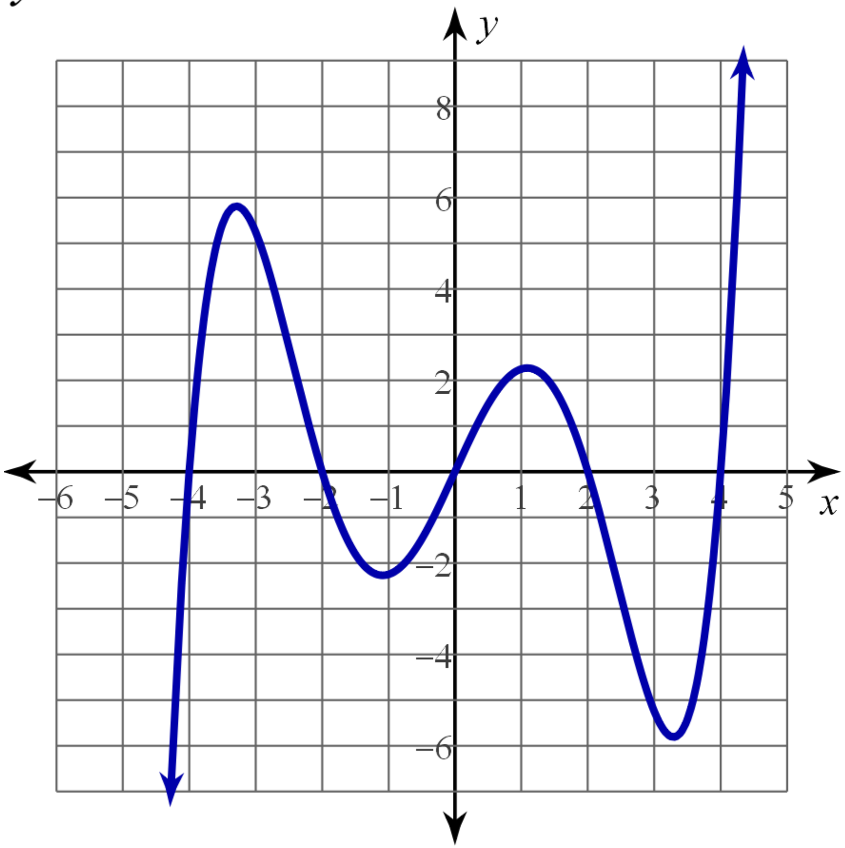
→ At the right and left ends of the graph (where the input value is a *gigantic* positive or negative number) the largest power in the polynomial has the largest effect on the output value (y-value) so this term dominates the effect on end-behavior.

Polynomial Degree → End Behavior?

$y = x^4 - 4x^2 + 1$



$y = x^5 - x^4 - 4x^2 + 1$

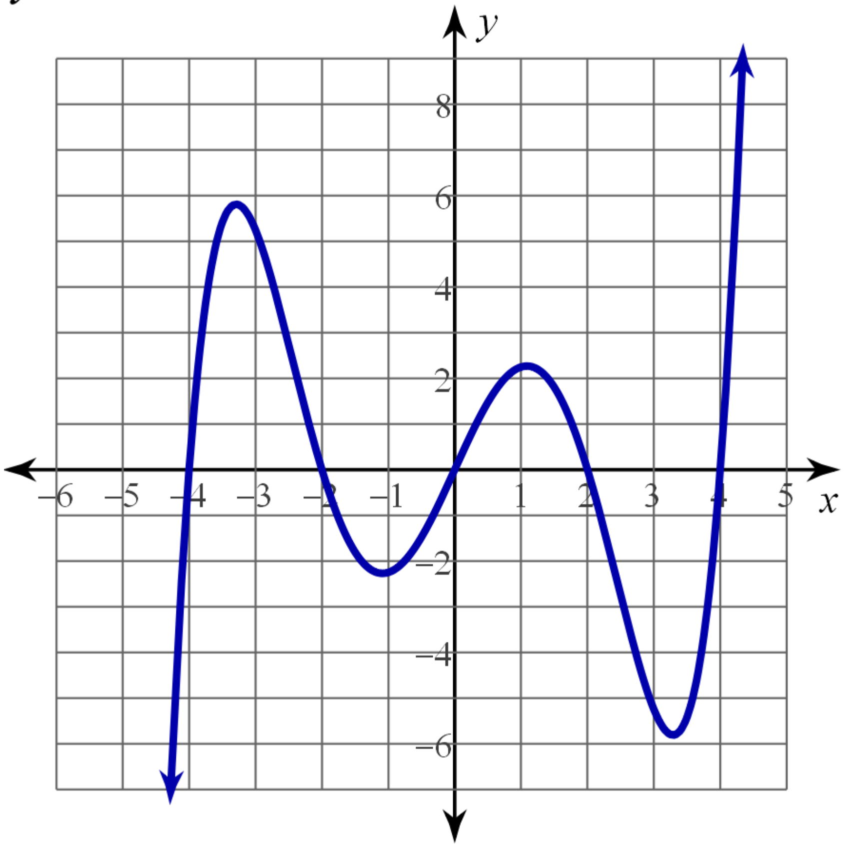


All <u>even degree</u> polynomials have the <u>same end behavior!</u>

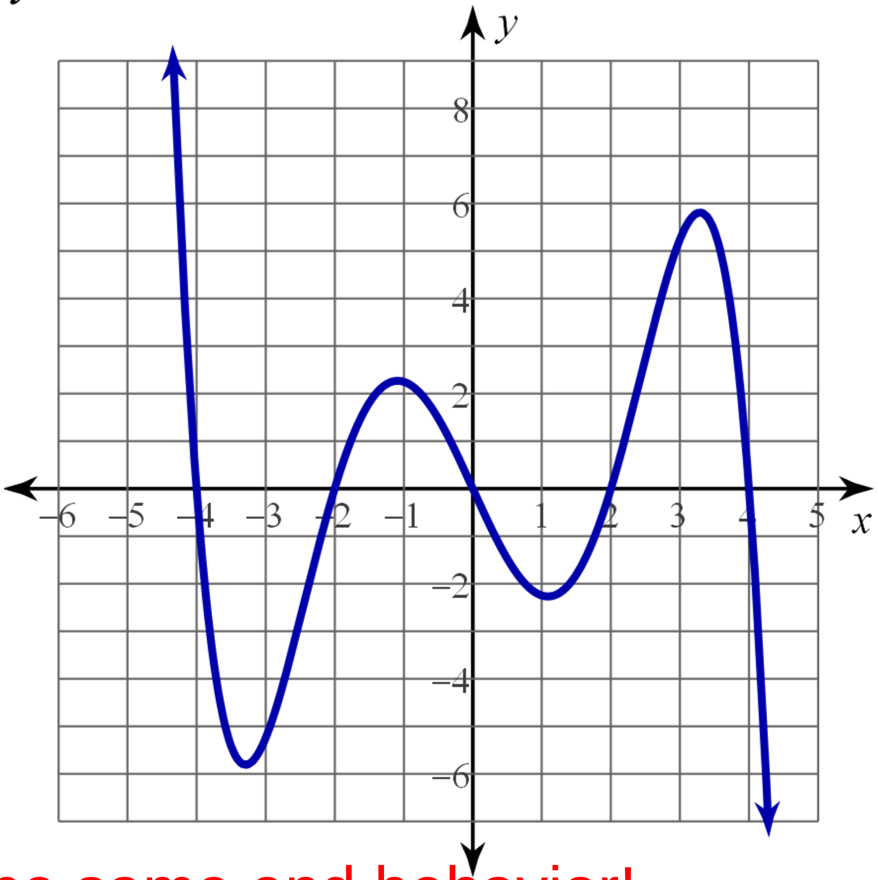
All <u>odd degree</u> polynomials have the <u>same end behavior!</u>

Lead Coefficient & Degree→ End Behavior?

$y = x^5 - x^4 - 4x^2 + 1$



$y = -x^5 - x^4 - 4x^2 + 1$



All odd degree polynomials have the same end behavior!

negative lead coefficient: reflection across the x-axis, all negative-odd polynomials have the same end behavior!

4^{th} degree polynomial → "4 zeroes"

$$y = (x - 1)(2x + 8)(x - 3)(x + 2)$$

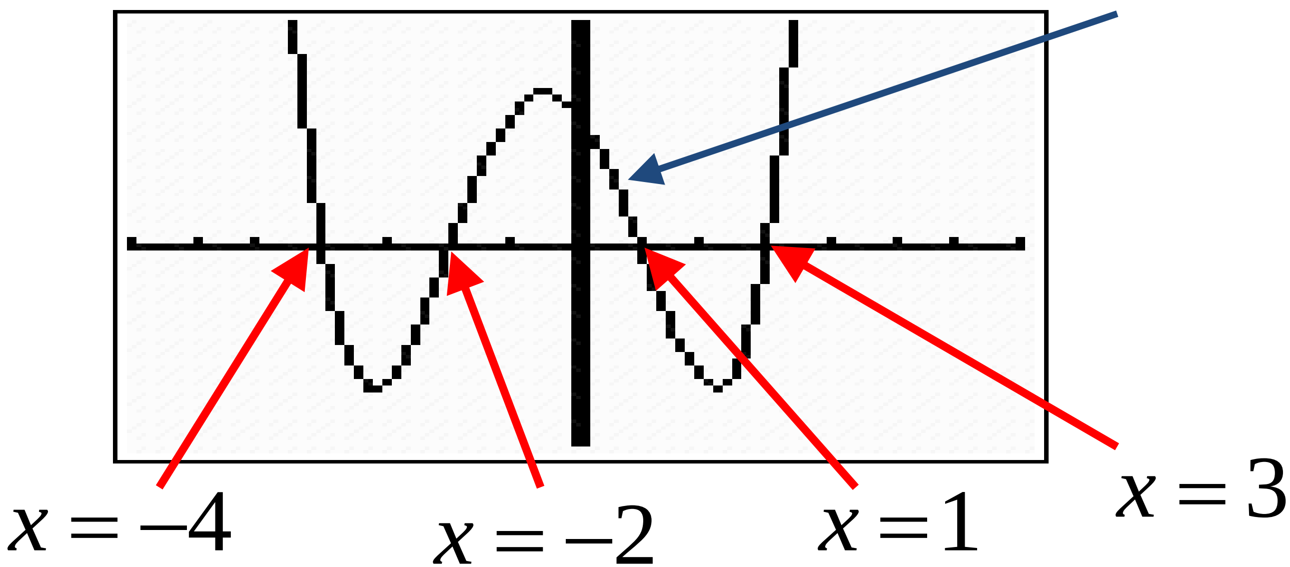
Lead term: "left*left*left*left" $= 2x$

$$y = (x - 1)(2x + 8)(x - 3)(2x + 2)$$

Constant term: (y-intercept) "right * right * right * right" = ? $= 48$

y-intercept = 48

$$y = 2x + \ldots(\text{other terms})\ldots + 48$$



Complex Conjugates Theorem

If f(x) is a polynomial and if (x + bi) is a factor (–bi is a zero) then its complex conjugate, (x - bi) is also a factor (and +bi is a zero) of f(x).

Example: $0 = x^2 + 4 \rightarrow 0 = (x - 2i)(x + 2i)$

x = 2i, x = 2i

Example: $0 = x^4 + 5x^3 + 13x^2 + 45x + 36$

$$0 = (x + 4)(x + 1)(x - 3i)(x + 3i)$$

x = -4, -1, *3i*, *-3i*

Irrational Roots Theorem

If f(x) is a polynomial and if $(x-\sqrt{b})$ is a factor of the polynomial ($\rightarrow \sqrt{b}$ is a zero) then its irrational conjugate $(x+\sqrt{b})$ is also a factor of the polynomial ($\rightarrow \sqrt{b}$ is also a zero).

Example: $0 = x^2 - 3 \quad \rightarrow 0 = (x-\sqrt{3})(x+\sqrt{3})$

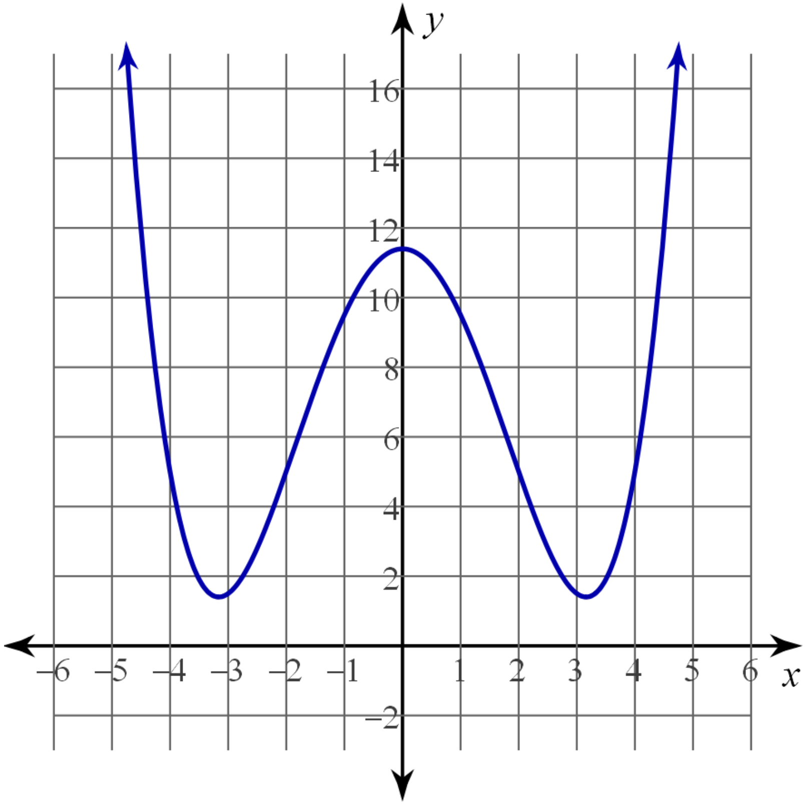
$$x = \sqrt{3}, \qquad -\sqrt{3}$$

Example: $0 = x^4 - x^2 - 20$

$$\rightarrow 0 = (x+2i)(x-2i)(x-\sqrt{5})(x+\sqrt{5})$$

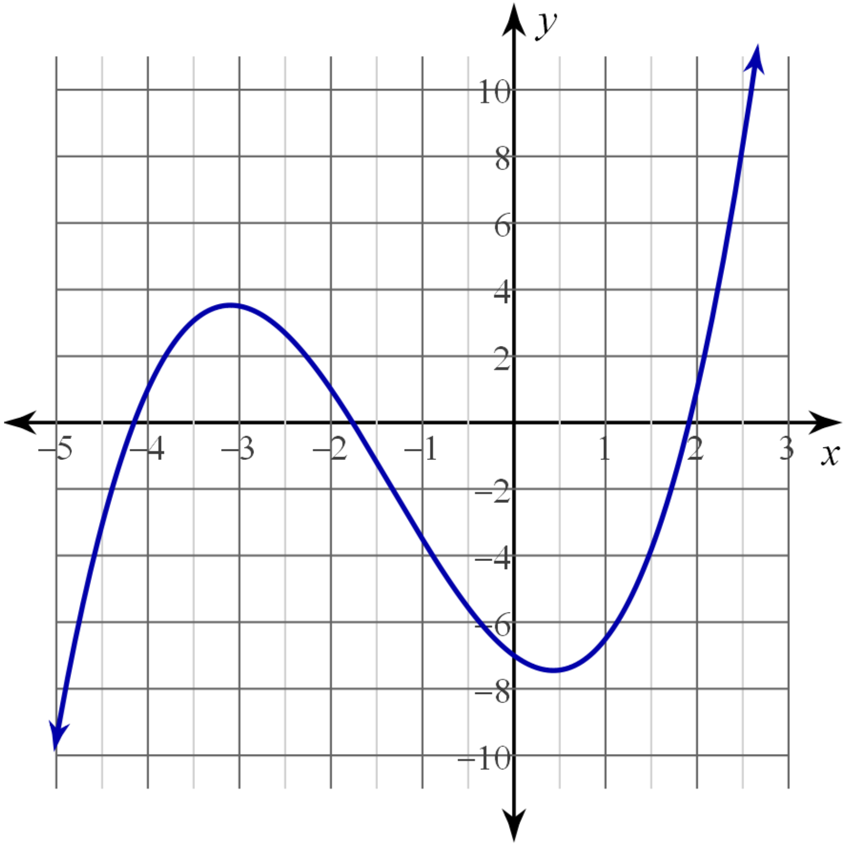
$$x = -2i, \qquad 2i, \qquad \sqrt{5}, \qquad -\sqrt{5}$$

Does an <u>even</u> degree polynomial necessarily cross the x-axis?



All zeroes can be imaginary for even-degree polynomials.

Does an <u>odd</u> degree polynomial necessarily cross the x-axis?



Since the end-behavior is down left/ up right, <u>*it must cross the x-axis*</u> so it has <u>*at least one real zero.,*</u>

Describe the $$f(x) = -8x^2 + 14x + 8$$

(1) end behavior Negative, even degree → down left/right

(2) number of real zeroes and/or imaginary zeroes

Make a table of the possible zeroes by category

Degree	Real zeroes	Imaginary Zeroes
2	2	0
	1 (mult 2)	0
	0	2

Describe the $f(x) = 2x^3 - 5x^2 + 14x + 8$

(1) end behavior positive, odd degree → down left, up right

(2) number of real zeroes and/or imaginary zeroes

Degree	Real zeroes	Imaginary Zeroes	
3	0	3 Not possible	Why not?
	1	2	
	1 mult 2	1 Not possible	Why not?
	1 mult 3	0	
	2	1 Not possible	Why not?
	1, 1 mult 2	0	
	3	0	

Since “multiplicities” are counted separately, it’s easier to just count the number of zeroes without specifying them as multiplicities.

Degree	Real zeroes	Imaginary Zeroes
3	0	3 Not possible
	1	2
	2	1 Not possible
	3	0

Describe the
(1) end behavior

$$f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8$$

Positive, even degree   Up on left/right

(2) number of real zeroes and/or imaginary zeroes

Degree	Real zeroes	Imaginary Zeroes
4	0	4
	1	3 Not possible
	2	2
	3	1 Not possible
	4	0