S14 Exponential Growth and Decay

1. According to the U.S. Census Bureau, in 2012 the city of Georgetown, Texas, a suburb of Austin was one of the fastest growing cities in the entire nation. In 2012, the population of Georgetown was 52,303.

   a) Assuming that the population increases at a constant percent rate of 3%, determine the population of Georgetown (in thousands) in 2013.

   b) Determine the population of Georgetown (in thousands) in 2014.

   c) Divide the population in 2013 by the population in 2012 and record this ratio.

   d) Divide the population in 2014 by the population in 2013 and record this ratio.

   e) What do you notice about the ratios in parts c and d? What do these ratios represent?

Linear functions represent quantities that change at a constant average rate (slope). Exponential functions represent quantities that change at a constant PERCENT rate.

Population growth, sales and advertising trends, compound interest, spread of disease, and concentration of a drug in the blood are examples of quantities that increase or decrease at a constant percent rate.

2. a) Let t represent the number of years since 2012 (t=0 corresponds to 2012). Use the results from problem 1 to complete the following table.

t, years (since 2012)	0	1	2	3	4	5
P, population (in thousands)						

Once you know the growth factor, b, and the initial value, a, you can write the exponential equation. In this situation, the initial value is the population in thousands in 2012 (t=0), and the growth factor is b = 1.03.

   b) Write the exponential equation, $P = a \cdot b^t$, for the population of Georgetown, TX.

Example 2: Recall that a percent increase can be determined in different ways.

Method 1:
A common method for calculating an amount after a percent increase is to determine the amount of increase and add to the original amount.

For example, we pay an increased percent for commodities in Utah due to an average 6.5% sales tax.

To determine the increased cost of a pair of $30 jeans due to sales tax, we might first determine the amount of the 6.5% increase, then add it to the base amount of $30.

30(.065)= $1.95 (amount of tax)

1.95 + 30 = $31.95

Method 2:
Another way to determine an amount after a percent increase is to first determine the total percent after the increase.

For example, the $30 price of the jeans could be considered 100% of the base amount, so the percent we would pay after the 6.5% tax increase would be 106.5%.

Then we would determine 106.5% of 30.

30(1.065) = $31.95

<u>In this case, 1.065 is considered the GROWTH FACTOR because we multiply by this number to get the amount that results from a 6.5% rate increase.</u>

The same reasoning can be applied for percent decreases.

To calculate an amount after a percent decrease, we can first determine the resulting percent after the decrease.

For example, a $40 coat might be on sale for 20% off. $40 would be 100% of the price, so the sale price AFTER the 20% decrease would be 80% of the original price.
Then determine 80% of $40 to determine the amount you would pay for the coat on sale.

40(.80) = $32

<u>In this case, .80 could be considered the DECAY FACTOR because we multiply by this number to get the amount that results from a 20% decrease.</u>

3. Determine if the sale price of the coat determined in the example above is the same sale price found by calculating 20% of 40 and then subtracting that amount from 40.

4. a) Use Method 2 from Example 2 to calculate the amount you would pay for a $65 online order after paying an additional 15% charge for shipping.

   b) What is the GROWTH FACTOR you used in part (a)?

5. a) Compared to traditional incandescent light bulbs, energy-efficient light bulbs such as halogen incandescents, compact fluorescent lamps (CFLs), and light emitting diodes (LEDs) use about 25%-80% less energy.

   Use Method 2 from Example 2 to calculate the amount you would pay for your electric bill if you could decrease your payment by 25% by upgrading to energy-efficient light bulbs. (Assume that previous electric bills were $150.)

   b) What is the DECAY FACTOR you used in part (a)?

   c) What is the relationship between the decay factor and the percent decrease?

6. What is the growth factor for a growth rate of 8%?

7. What is the growth RATE for a growth factor of 1.054?

8. a) Complete the following table.

t	Calculation for population (in thousands)	Exponential Form	P(t), Population in thousands
0	52.3	$52.3(1.03)^0$	
1	(52.3) 1.03	$52.3(1.03)^1$	
2	(52.3)(1.03)(1.03)		
3			

b) Use the pattern in the table in part (a) to help you write the equation for P(t), the population of Georgetown (in thousands), using t, the number of years since 2012, as the input value. How does your result compare with the equation obtained in problem #2b?

c) Determine the growth factor for the function P(t).

d) Determine the growth RATE of the population of Georgetown written as a percent.

e) Determine P(8). What is the practical meaning of the value you found for P(8)?

f) Graph the function with your graphing calculator. Use the window Xmin = 0, Xmax= 100, Ymin = 0, Ymax = 1000. Determine P(0) from your graph. What is the graphical and practical meaning of P(0)?

g) Use your graph to predict the population of Georgetown, Texas, in 2022. Then use the graph to determine the year that the population will reach 75,000, assuming it continues to grow at the same rate. Remember that P(t) is the number of thousands of people.

h) Predict the population of Georgetown in 2035? Do you think this is an accurate prediction? Why or why not?

9. Assuming the growth rate remains constant, how long will it take the population of Georgetown, Texas, to double its 2012 population? Explain how you reached this conclusion.

10. Determine the growth factor and the growth rate of the function defined by $f(x) = 250(1.7)^x$.

11. You are working at a wastewater treatment facility. You are presently treating water contaminated with 18 micrograms (μg) of pollutant per liter. Your process is designed to remove 20% of the pollutant during each treatment. Your goal is to reduce the pollutant to less than 3 micrograms per liter.

a) What percent of the pollutant present at the start of a treatment remains at the end of the treatment?

b) The concentration of pollutants is 18 micrograms per liter at the start of the first treatment. Use the result of part (a) to determine the concentration at the end of the first treatment.

c) Complete the following table. Round the results to the nearest tenth.

n, Number of Treatments	0					
C(n) Concentration of Pollutant at end of nth treatment						

d) Write an equation for the concentration, C(n), of the pollutant as a function of the number of treatments, n.

12. Determine the decay factor and the decay rate of the function defined by $f(x) = 123(0.43)^x$.

13. If the decay rate of a function is 5%, determine the decay factor.

14. If the decay rate is 2.5%, what is the decay factor?

Congratulations! You have inherited $20,000! Your grandparents suggest that you use half of the inheritance to start a retirement fund. Your grandfather claims that an investment of $10,000 could grow to over half a million dollars by the time you retire. You are intrigued by this statement and decide to investigate whether this can happen.

15. a) Suppose the $10,000 is deposited in a bank at 3.5% annual interest. Use a growth factor to find the amount in the account after one year.

b) Using the amount in the account after one year as the starting amount, compute the amount in the account after the 2^{nd} year. (Use the growth factor again.)

c) Write an exponential equation using the starting value in the account and the growth factor.

This is an example of compounded interest because during the 2nd year and any years to follow, interest will be earned off the interest that was earned previously. In this example, the interest is compounded annually but interested can be compounded at other fixed intervals. Often it is compounded quarterly (4 times a year), monthly (12 times a year), or daily (365 times a year).

If interest is compounded, the current balance is given by the formula

$$A = P(1 + \frac{r}{n})^{nt}$$

where A is the current balance, or compound amount in the account,
P is the principle (the original amount deposited),
r is the annual interest rate (in decimal form)
n is the number of times per year that interest is compounded,
and *t* is the time in years the money has been invested.
The given formula is called the **compound interest formula.**

16. How does the compound interest formula compare to the equation you found in problem #15c?

17. Determine how much money you would have in your account if you invested your $10,000 at the same annual interest rate of 3.5% but the interest is compounded daily instead of yearly. How does this amount compare to the amount you would have after 2 years of compounding annually (problem #15b)?

Compounding more often at the same rate will yield a greater amount over a given time. In fact, you could calculate the amount when interest is compounded every hour, minute, or even second. However, compounding more frequently than every hour doesn't increase the balance very much because the growth factor doesn't change much as n gets larger and larger.

So for compounding that occurs more often than daily compounding, banks will use a different formula for what is called "**continuous**" compounding:

$$A = Pe^{rt}$$

where A is the current amount, or balance, in the account;
P is the principal;
r is the annual interest rate (annual percentage in decimal form)
t is the time in years that your money has been invested; and
e is the base of the continuously compounded exponential function (*e* is called Euler's constant and is approximately 2.718)

18. Determine how much money you would have in your account after 2 years if you invested your $10,000 at the same annual interest rate of 3.5% compounded **continuously.** How does this compare to the amounts in #15b (compounded annually) and #17 (compounded daily)?

19. Historically, investments in the stock market have yielded an average rate of 11.7% per year. Suppose you invest $10000 in an account at 11% annual interest rate that compounds continuously.

    a) Use the appropriate formula to determine the balance after 35 years.

    b) What is the balance after 40 years?

    c) Your grandfather claimed that $10,000 could grow to more than half a million dollars by the time you retire (in 40 years). Is your grandfather correct in his claim?

A General Formula for Continuous Growth

Considering the equation for continuous growth of money in a bank account ($A = Pe^{rt}$), a more general form of this equation is used for other types of continuous growth. There are many other situations in which growth occurs continuously and not just yearly, monthly, or daily. Population growth is a good example of continuous growth because babies are born every second of every day, not just on a monthly or yearly schedule.

Whereas $A = Pe^{rt}$ is used when dealing with the continuous growth of money, a more general formula for continuous growth or decay is: $y = ae^{kt}$
where A has been replaced with *y*, the output;
P has been replaced with a, the initial value;
And *r* has been replaced with *k*, the continuous growth or decay rate.

These continuous growth or decay equations are closely related to basic exponential equations.

For example:
The equation $y = 42(1.23)^t$ can be written as the continuous growth equation $y = 42e^{.207t}$. The difference between the two equations is that the first reflects the growth at a rate of 23% over a number of years, *t*, in which the growth increases once each year. (If *t* represented hours instead of years then the percent growth would occur once per hour, etc.)
The second one, the continuous growth equation, reflects continuous, ongoing growth throughout the years at a ***continuous growth rate*** of 20.7%. Both equations will give the same approximate value for y for the same number of years. Notice that the rate for continuous growth is slightly less then the rate for annual growth. The continuous growth rate, k, that corresponds to a growth factor, b, can be found by solving for *k* in the equation $b = e^k$. Because the value of *k*= .207 represents a continuous ***growth*** rate, k is a positive number.

The equation $y = 35(.97)^t$, has a decay rate of 1- .97 = .03 or 3%; this equation can be written as the continuous exponential decay equation,
$y = 35e^{-.03t}$. Notice that *k* is negative when representing a rate of decay.

Identify the given exponential functions as increasing or decreasing. In each case, give the initial value and rate of increase or decrease.

a) $R(t) = 25(1.098)^t$

b) $S = 3025(0.72)^t$

c) $f(x) = 95.2e^{-.04x}$

d) $B = 0.59e^{.076x}$