Math-1060
Lesson 7-2

Dot Product of Vectors

Vector Addition $v = \langle 2,4 \rangle \quad u = \langle 5,1 \rangle$ $v + u = ?$

$$v + u = \langle 2+5, 4+1 \rangle = \langle 7,5 \rangle$$

Vector Multiplication has three forms

1) Dot Product of 2 Vectors

2) Scalar Multiplication

3) Cross Product
(not covered in this book)

"Dot Product":

$$u = \langle x_1, y_1 \rangle \quad w = \langle x_2, y_2 \rangle$$

$$u \bullet w = x_1 x_2 + y_1 y_2$$

$$m = \langle 2,-3 \rangle \quad n = \langle 4,2 \rangle$$

$$m \bullet n = (2*4) + (-3*2)$$

The "Dot Product" of two vectors results in a *scalar.*

m•n = 2

"M dot N equals 2."

$$n = \langle -3, -7 \rangle \qquad m = \langle -1, 4 \rangle$$

1. $m \bullet n = ?$

$$m \cdot n = (-3)(-1) + (-7)(4)$$

$$m \cdot n = 3 - 28$$

$$m \cdot n = -25$$

2. $n \bullet m = ?$

$$n \cdot m = (-1)(-3) + (4)(-7)$$

$$n \cdot m = 3 - 28$$

$$n \cdot m = -25$$

What did you learn here?

The dot product of vectors obeys the commutative law.

Do you remember what this is? $v = \langle 0,0 \rangle$

$m = \langle 3,4 \rangle$ $v \bullet m = ?$ $= (0 * 3) + (0 * 4)$

A vector dotted with the zero vector equals zero

Other Dot Product Properties

$$m = \langle 2,4 \rangle \qquad v = \langle -1,3 \rangle \qquad w = \langle 5,-1 \rangle$$

$$m \bullet (v + w) = ? \quad = m \bullet \langle (-1 + 5,)\ (3 - 1) \rangle \quad = m \bullet \langle 4, 2 \rangle$$

$$= (2 * 4) + (4 * 2) \quad \boxed{= 16}$$

$$(m \bullet v) + (m \bullet w) = ?$$

$$= [(2 * -1) + (4 * 3)] + [(2 * 5) + (4 * -1)]$$

$$= (-2 + 12) + (10 - 4) \quad \boxed{= 16}$$

What did you learn here?

Vector Dot products follow the Distributive Property

We can also show that: $(c * v) \bullet w = v \bullet (c * w) = c(v \bullet w)$

Scalar multiplication and the dot products of vectors is also commutative (order doesn't matter).

Do you remember what this means? $|m| = ?$

$$m = \langle 3,4 \rangle$$

(magnitude of the vector m)

<u>Your turn:</u>

$$|m|^2 = ?$$

$$= \left(\sqrt{3^2 + 4^2}\right)^2 = 25$$

$$m \bullet m = ?$$

$$m \cdot m = (3)(3) + (4)(4)$$

$$m \cdot m = 25$$

What did you learn here?

A vector dotted with itself equals the square of its magnitude.

<u>OR</u>: Magnitude of a vector = sq root of its own dot product

$$|m| = \sqrt{m \cdot m}$$

Dot Product Properties

1. DP’s are commutative (order doesn’t matter)

2. The zero vector dotted with any vector = 0

3. DP’s follow the distributive property of multiplication over addition.

4. Scalar multiplication and DP’s are commutative.

5. The square root of the dot product of a vector taken with itself = magnitude of the vector.

Use dot products to find the magnitude of the vector ‘m’

$$m = \langle 5,6 \rangle$$

$$|m| = \sqrt{m \cdot m}$$

$$|m| = \sqrt{5^2 + 6^2}$$

$$|m| = \sqrt{25 + 36}$$

$$|m| = \sqrt{61}$$

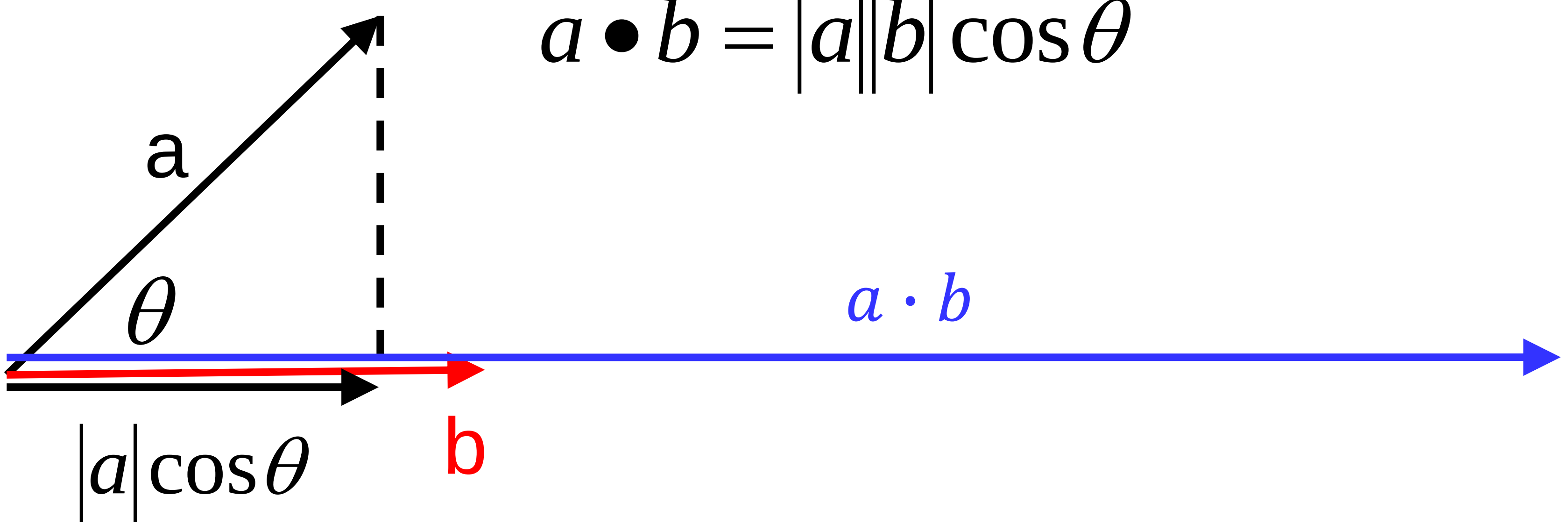
What does the “dot product” mean, physically?

Dot product measures how well two vectors are “aligned” with each other.

Dot product is zero when the vectors are perpendicular, and has maximum magnitude when they are parallel.

If they are perfectly aligned, the value of the scalar product is the product of the magnitudes of the two vectors, and for less and less parallel vectors, the scalar product goes down.

$$a \bullet b = |a||b|\cos\theta$$



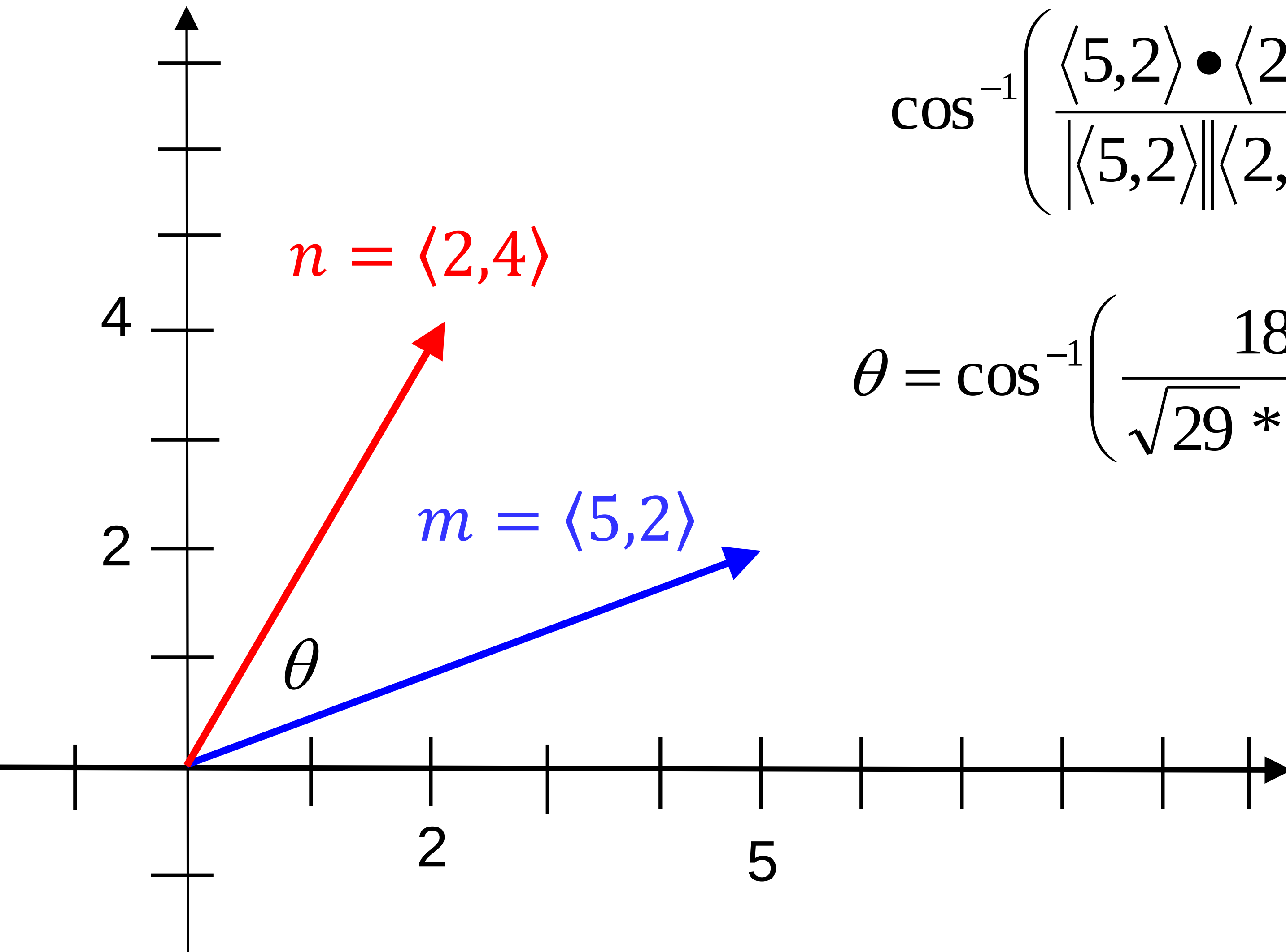
Finding the <u>Angle</u> between 2 Vectors

$$a \bullet b = |a||b|\cos\theta$$

$$\cos\theta = \frac{m \bullet n}{|m||n|} \qquad \cos^{-1}\left(\frac{m \bullet n}{|m||n|}\right) = \theta$$

$$\cos^{-1}\left(\frac{\langle 5,2\rangle \bullet \langle 2,4\rangle}{|\langle 5,2\rangle||\langle 2,4\rangle|}\right) = \theta$$

$$\theta = \cos^{-1}\left(\frac{18}{\sqrt{29} * \sqrt{20}}\right) = 41.6^\circ$$



$$a \bullet b = |a||b|\cos\theta$$

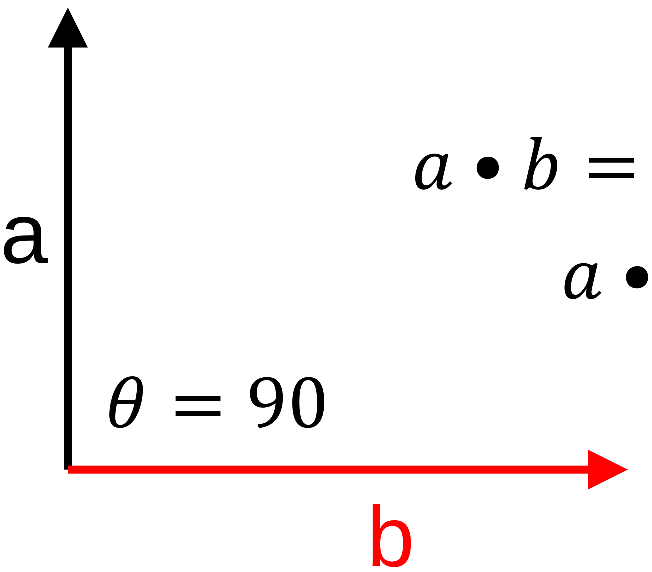
$$\cos^{-1}\left(\frac{m \bullet n}{|m||n|}\right) = \theta$$

$$m = \langle -4, -3 \rangle$$

$$n = \langle -1, 5 \rangle$$

Find the angle between the two vectors ‘m’ and ‘n’.

$$\theta = 115.6^\circ$$



$$a \bullet b = |a||b|\cos 90$$

$$a \bullet b = 0$$

What did you learn here?

If two vectors are perpendicular to each other, their dot product equals zero.

Orthogonal: means perpendicular.

Since the zero vector has no direction, it would be hard to say it is perpendicular to any vector.

BUT, we can say it is orthogonal to every vector.

So, except for the zero vector orthogonal = perpendicular

Determining of Vectors are Orthogonal

If $m \bullet v = 0$ then vectors 'm' and 'n' are orthogonal.

If vectors 'm' and 'n' are orthogonal then $m \bullet v = 0$

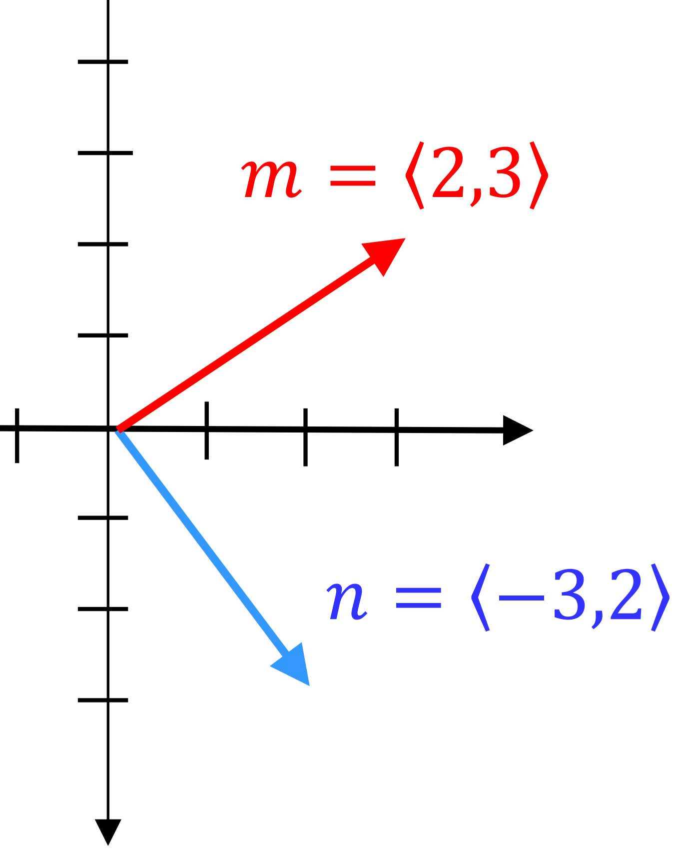
Vectors 'm' and 'n' are orthogonal if and only if $m \bullet v = 0$

(Both the conditional statement and its converse are true.)

Are the vectors orthogonal? $m = \langle 2,3 \rangle$ $n = \langle -3,2 \rangle$

$$m \bullet n = (2)(-3) + (3)(2) = 0$$

Since $m \bullet n = 0$ the vectors are orthogonal.



Are vectors ‘v’ and ‘w’ orthogonal?

v = -3*i* + 4*j* w = 20*i* + 15*j*

$$v \bullet w = (-3)(20) + (4)(15)$$

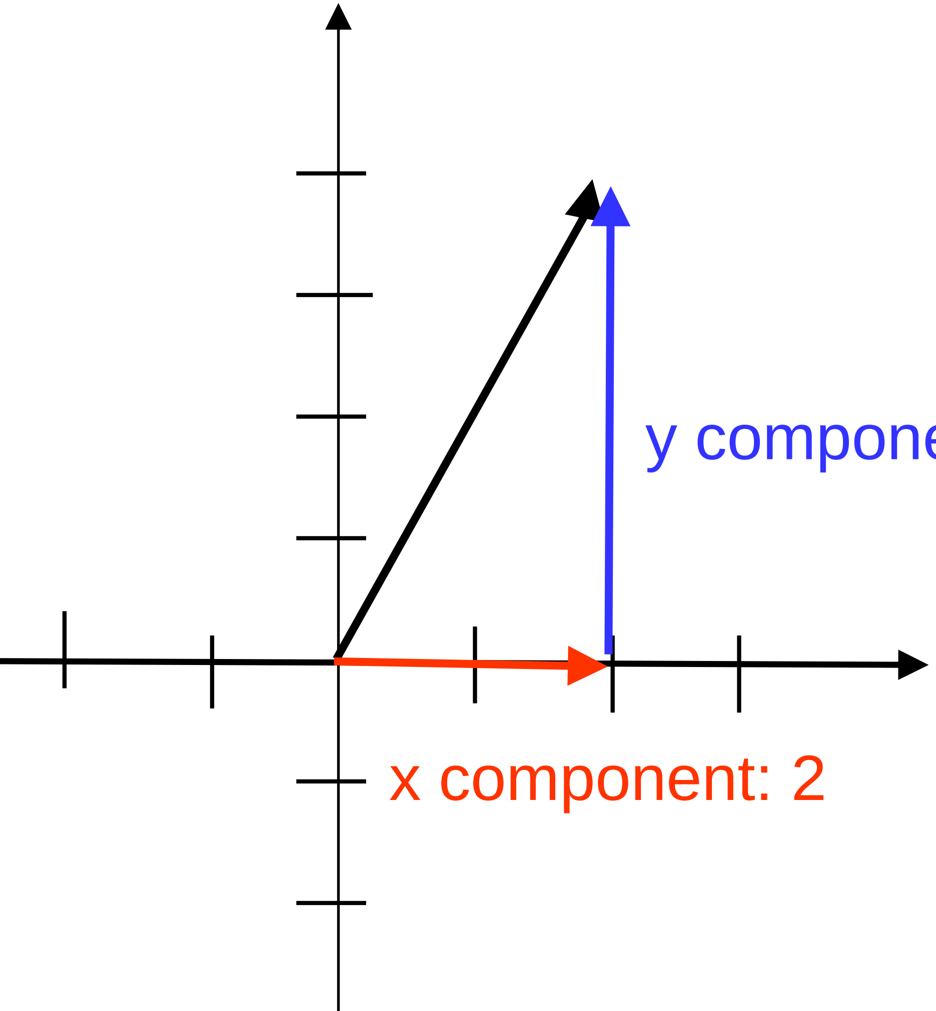
$$v \bullet w = -60 + 60$$

YES, ‘v’ and ‘w’ are orthogonal.

Remember this?

(Resolving a vector into its horizontal and vertical components)

$$v = \langle 2,4 \rangle = 2i + 4j$$

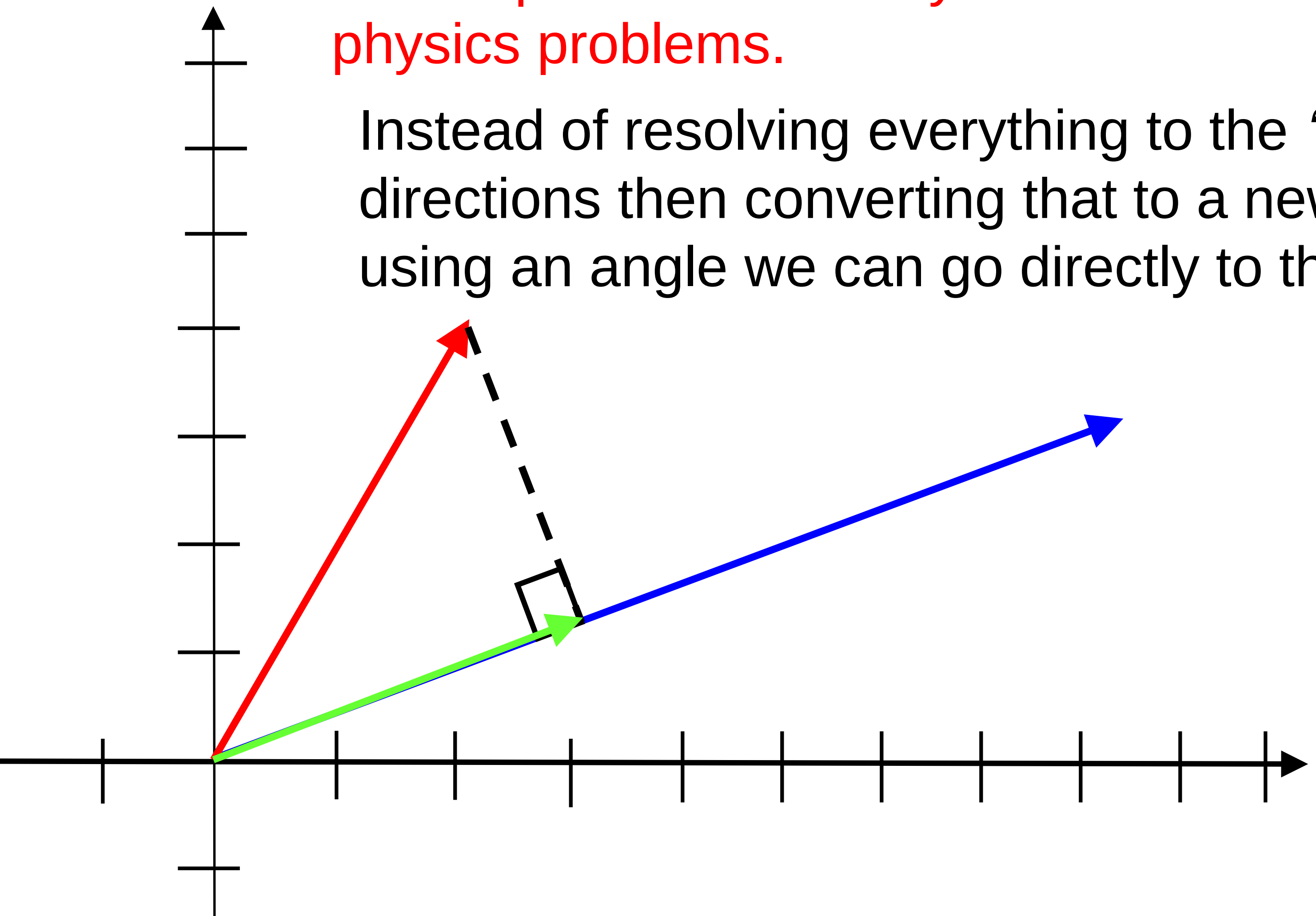


“*Projection*” of one vector onto another

Determines the component of one vector that is in the *same direction* as another vector.

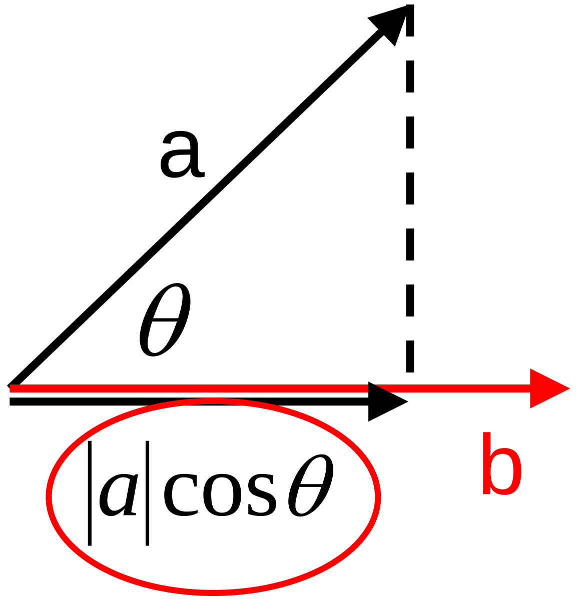
This helps us more easily find resultant forces in physics problems.

Instead of resolving everything to the *‘i’* and *‘j’* directions then converting that to a new direction using an angle we can go directly to the answer.



Projecting a Vector onto Another Vector.

The component of vector ‘a’ in the direction of vector ‘b’ is the magnitude of ‘a’ times the cosine of the angular difference between the two vectors.



Using the definition of the dot product, we can find cos(ϴ)

$$a \bullet b = |a||b|\cos\theta$$

The “Dot Product” of two vectors results in a *scalar.*

$$\frac{a \bullet b}{|b|} = |a| * \cos\theta$$

Rearranging we get (still a scalar):

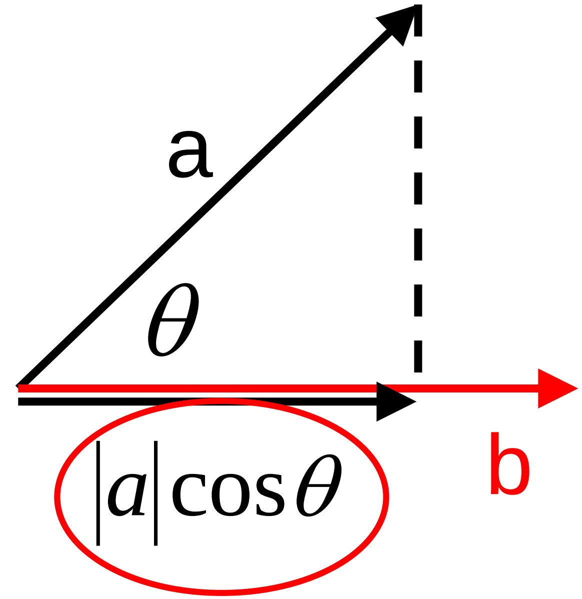
Now we multiply this scaler times the unit vector ‘b’ (gives magnitude and direction of vector ‘a in the direction of ‘b’).

$$u_b = \frac{b}{|b|}$$

Projecting a Vector onto Another Vector.

$$a \bullet b = |a||b| \cos\theta$$

Definition of a Dot Product



$$\frac{a \bullet b}{|b|} = |a| * \cos\theta$$

Rearranged (still a scalar):

Now we multiply this scaler times the unit vector 'b' (gives magnitude and direction of vector 'a in the direction of 'b').

$$u_b = \frac{b}{|b|} \qquad \frac{a \bullet b}{|b|} * u_b = (|a| * \cos\theta) * u_b$$

$$\frac{b}{|b|} * \frac{a \bullet b}{|b|} = (|a| * \cos\theta) * \frac{b}{|b|}$$

$$proj_b a = \left(\frac{a \bullet b}{|b|^2}\right) b$$

“*Projection*” ‘m’ in the direction of ‘n’

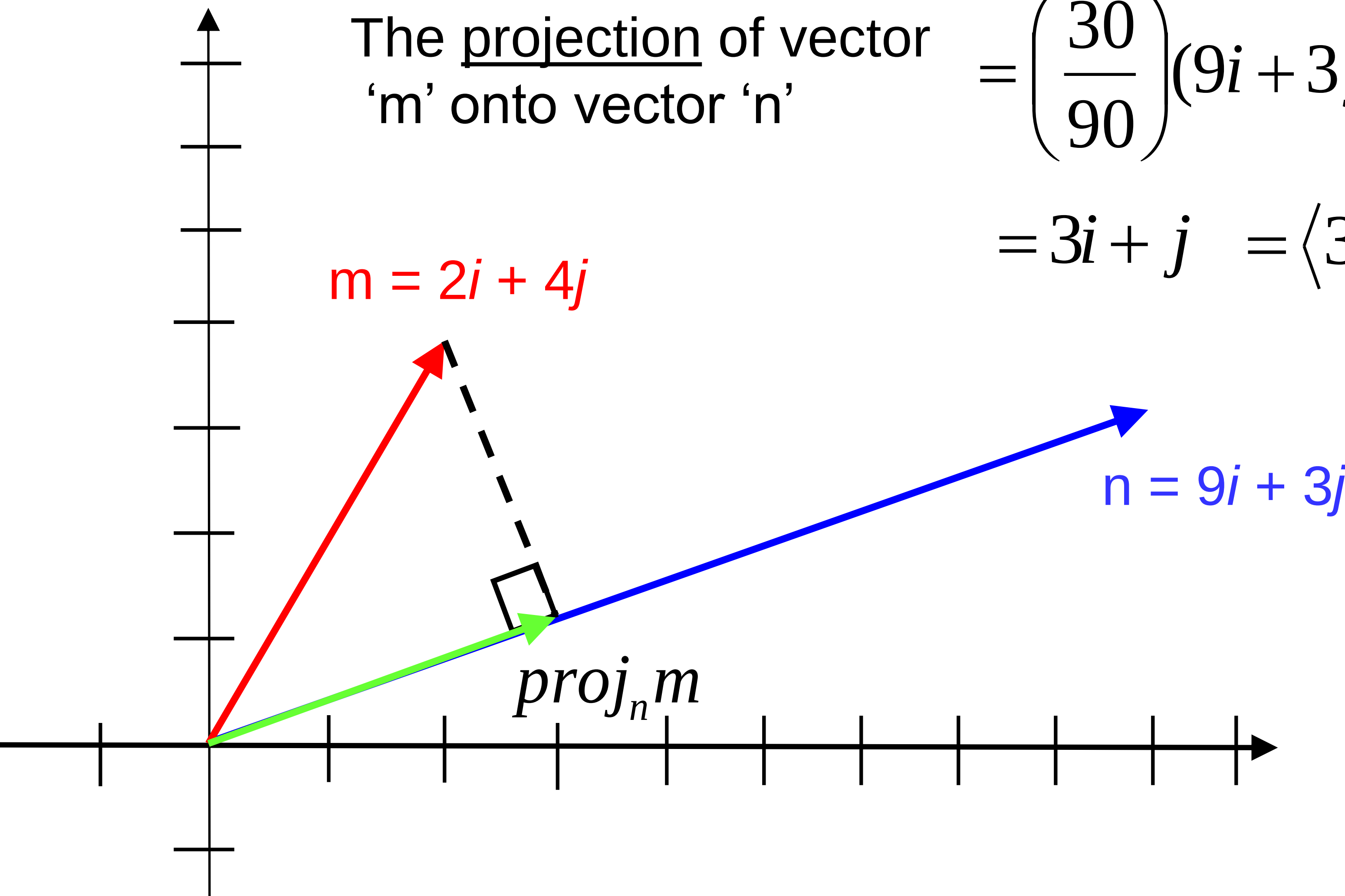
$$proj_n m = \left(\frac{m \bullet n}{|n|^2} \right) n$$

$$= \left(\frac{18+12}{81+9} \right)(9i+3j)$$

The projection of vector ‘m’ onto vector ‘n’

$$= \left(\frac{30}{90} \right)(9i+3j)$$

$$= 3i + j \quad = \langle 3,1 \rangle$$



Find the projection of vector ‘v’ onto vector ‘w’.

$$proj_w v = \left(\frac{v \bullet w}{|w|^2} \right) w \qquad v = \langle -2,4 \rangle \quad w = \langle -6,2 \rangle$$

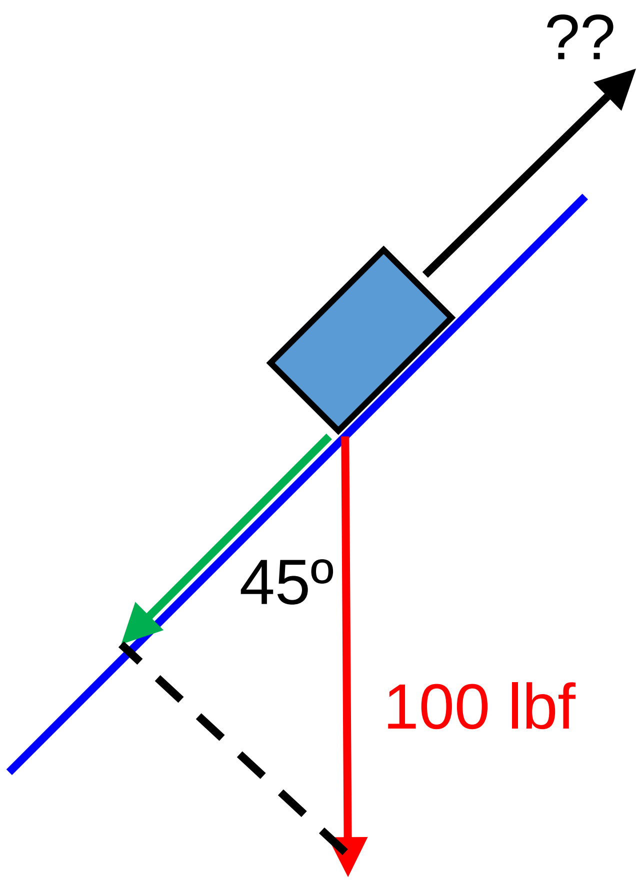
$$proj_w v = \frac{(-2 * -6) + (4 * 2)}{\left(\sqrt{(-6)^2 + 2^2}\right)^2} \langle -6,2 \rangle$$

$$proj_w v = \frac{20}{40} \langle -6,2 \rangle = \langle -3,1 \rangle$$

“Force” Problems

A box (on wheels → neglect friction) is on an incline of 45 degrees. The box weights 100 lbf. What force is required to keep the box from sliding down the hill?

(The question is asking: what is the projection of the vertical force of gravity along the direction of the slope?)

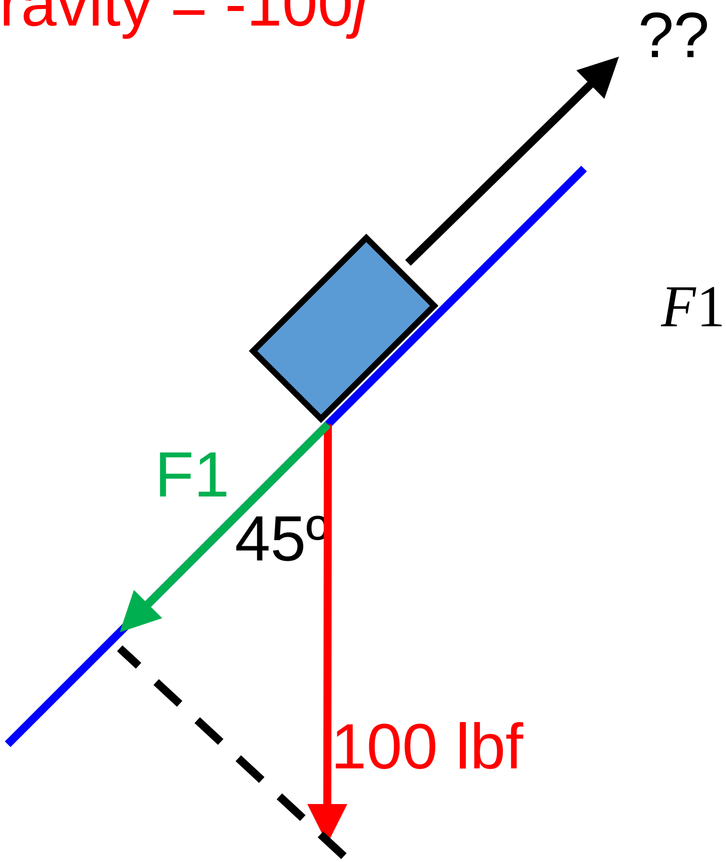


If the force pulling the box down the hill is opposed by a force of equal magnitude and opposite direction, then the box won’t move.

$$proj_{hill}gravity = \left(\frac{g \bullet h}{|h|^2} \right) h$$

Hill = cos 45º *i* + cos 45º *j*

Gravity = -100*j*



$$proj_{hill}gravity = \left(\frac{g \bullet h}{|h|^2} \right) h$$

$$F1 = \left(\frac{\langle 0, -100 \rangle \bullet \left\langle \sqrt{2}/2, \sqrt{2}/2 \right\rangle}{1^2} \right) \left\langle \sqrt{2}/2, \sqrt{2}/2 \right\rangle$$

$$F1 = -100 \left(\sqrt{2}/2 \right) \left\langle \sqrt{2}/2, \sqrt{2}/2 \right\rangle$$

$$F1 = -50i - 50j$$

Opposite force:

$$|F1| = \sqrt{50^2 + 50^2} = \sqrt{2 * 50^2} = 50\sqrt{2}$$

= 70.7 lbf

$$a \bullet b = |a||b| \cos \theta$$

Definition of a Dot Product

Work: A force acting through a distance.

$$Work = F \bullet dist$$

The work done by a force moving an object from point A to point B

Case 1: the force vector and the displacement (distance moved) vector are in *the same direction* (theta = 0)

$$work = \overrightarrow{F} \cdot \overrightarrow{AB} = |F|\left|\overrightarrow{AB}\right|$$

(magnitude * distance)

Units: foot-pounds

$$a \bullet b = |a||b|\cos\theta$$

$$\boxed{Work = F \bullet dist}$$

Work: A force acting through a distance.

The work done by a force moving an object from point A to point B

Case 2: the force vector and the displacement (distance moved) vector are in *different directions*.

Theta (the angle between the two vectors) is *not zero*

Example: A 10 lb force acting in the direction $1i + 2j$ moves an object 3 feet from (0, 0) to (3, 0)

Force has a magnitude AND a direction.

We must convert the direction vector of the force into a unit vector. The magnitude of the force times the unit vector of the direction of the force is the force vector.

Example: A 10 lb force acting in the direction 1*i* + 2*j* moves an object 3 feet from (0, 0) to (3, 0)

Force: magnitude and direction.

10 lbf is the magnitude. It acting in the same direction as 1*i* +2*j*.

The magnitude of the direction vector is: $|d| = \sqrt{1^2 + 2^2}$

$|d| = \sqrt{5}$ If we multiply the scalar (10 lbf) times the direction vector we'll get a force of $10\sqrt{5}$

We need a vector that is the same direction as: d = 1i + 2j but whose magnitude is '1'. We need the *unit vector* of vector 'd'

$$\overrightarrow{u_f} = \frac{\vec{d}}{|d|} = \frac{\langle 1,2 \rangle}{|\langle 1,2 \rangle|} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle \qquad \vec{F} = |F| * \overrightarrow{u_f}$$

$$Force = 10.\, lb_f \left(\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle \right)$$

Example: A 10 lb force acting in the direction $1i + 2j$ moves an object 3 feet from (0, 0) to (3, 0)

$$Work = F \bullet dist$$

$$force = \left(\left\langle \frac{10}{\sqrt{5}}, \frac{20}{\sqrt{5}} \right\rangle\right) lb_f$$

$$Work = \left(\left\langle \frac{10}{\sqrt{5}}, \frac{20}{\sqrt{5}} \right\rangle\right) lb_f \bullet \langle 3,0 \rangle \text{ ft}$$

$$Work = \left(\frac{10}{\sqrt{5}} * 3\right) + \left(\frac{20}{\sqrt{5}} * 0\right) ft * lb_f$$

$$Work = \left(\frac{30}{\sqrt{5}}\right) ft \bullet lb_f$$

$$Work \approx 13.4\, ft \bullet lb_f$$