

Math-1060

Lesson 6-6

Solving Trigonometric Equations

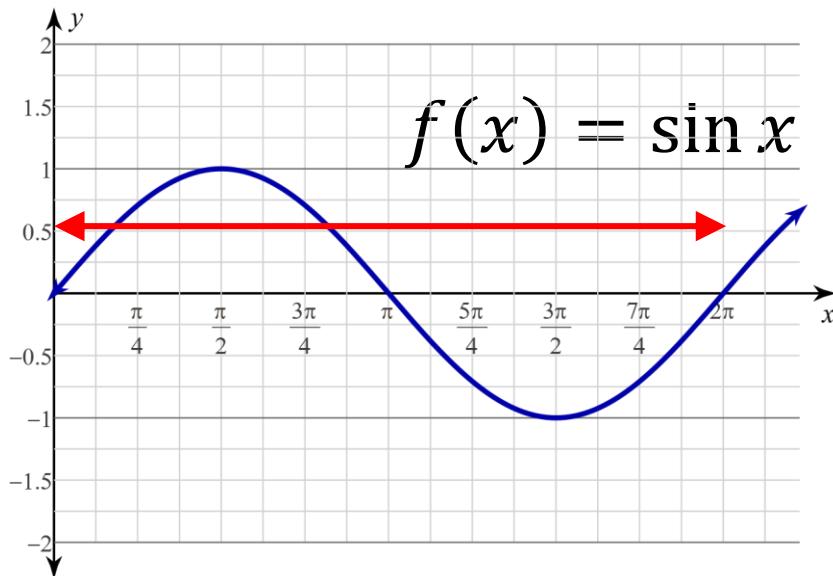
Solution: The value that, when substituted into the variable makes the equation “true.”

Solve: Using mathematical operations and properties to determine the solution to an equation.

Method 1: Solving by “inspection”.

Example: Solve the following equation on the interval $[0, 2\pi]$

$$\sin x = \frac{1}{2}$$



$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Or, you should know that

$$\sin 30 = \frac{1}{2}$$

The reference angle is 30° , so the solution is $30^\circ, 150^\circ$ (convert to radians).

Example: Solve the following equation on the interval $[0, 2\pi]$

$$\cos 2x = \frac{1}{2}$$

You should know that

$$\cos 60^\circ = \frac{1}{2} \quad 2x = 60^\circ$$

$$x = 30^\circ$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

On the interval $[0, 2\pi]$, solve:

$$\cos x = \frac{1}{2}$$

$$\cos 30^\circ = \frac{1}{2}$$

$$\sin 2x = \frac{1}{2}$$

$$\sin 30^\circ = \frac{1}{2}$$

$$2x = 30^\circ$$

$$x = 15^\circ$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{11\pi}{12}$$

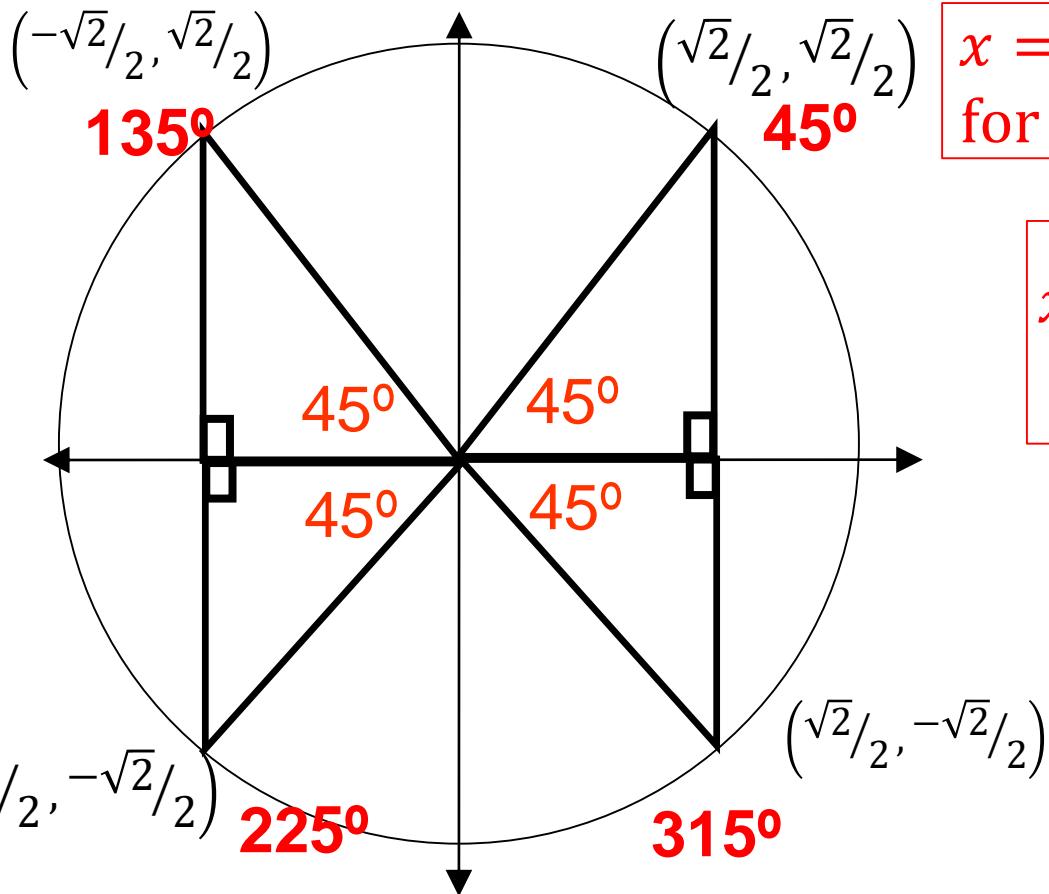
Solve the following equation on the interval $[0, 2\pi]$ $\sin x = \frac{\sqrt{2}}{2}$

You should know that

$$\sin 45^\circ = \frac{\sqrt{2}}{2} \quad x = 45^\circ$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

For $\Theta = (-\infty, \infty)$ solve: $\sin x = \frac{\sqrt{2}}{2}$ You must give *all solutions*.



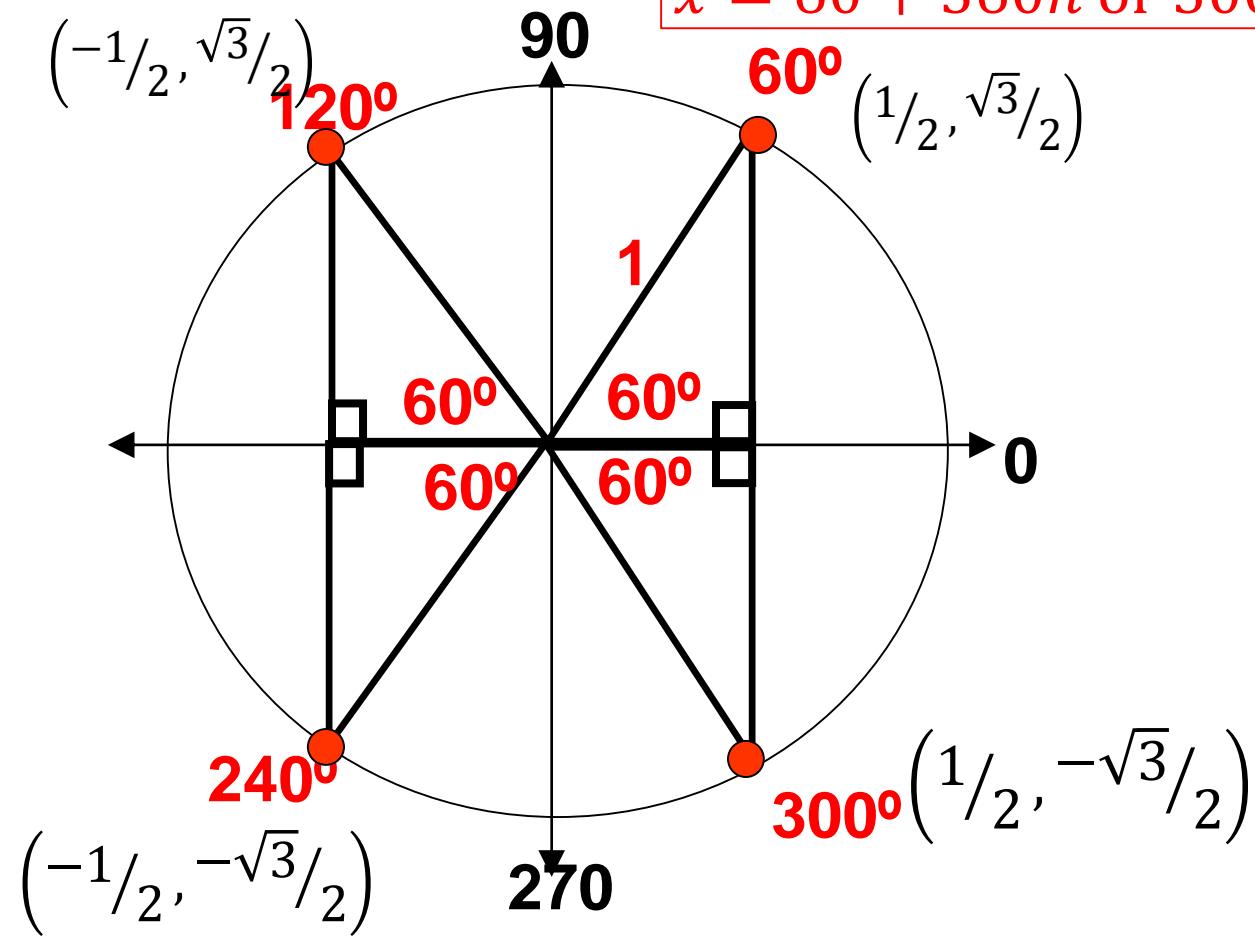
$$x = 45 + 360n \text{ or } 135 + 360n \text{ for } n \in \mathbb{Z}$$

$$x = \frac{\pi}{4} + 2\pi n \text{ or } \frac{3\pi}{4} + 2\pi n \text{ for } n \in \mathbb{Z}$$

For $\Theta = (-\infty, \infty)$ solve: $\sin x = \frac{\sqrt{3}}{2}$

$$x = \frac{\pi}{3} + 2\pi n \text{ or } \frac{2\pi}{3} + 2\pi n \text{ for } n \in \mathbb{Z}$$

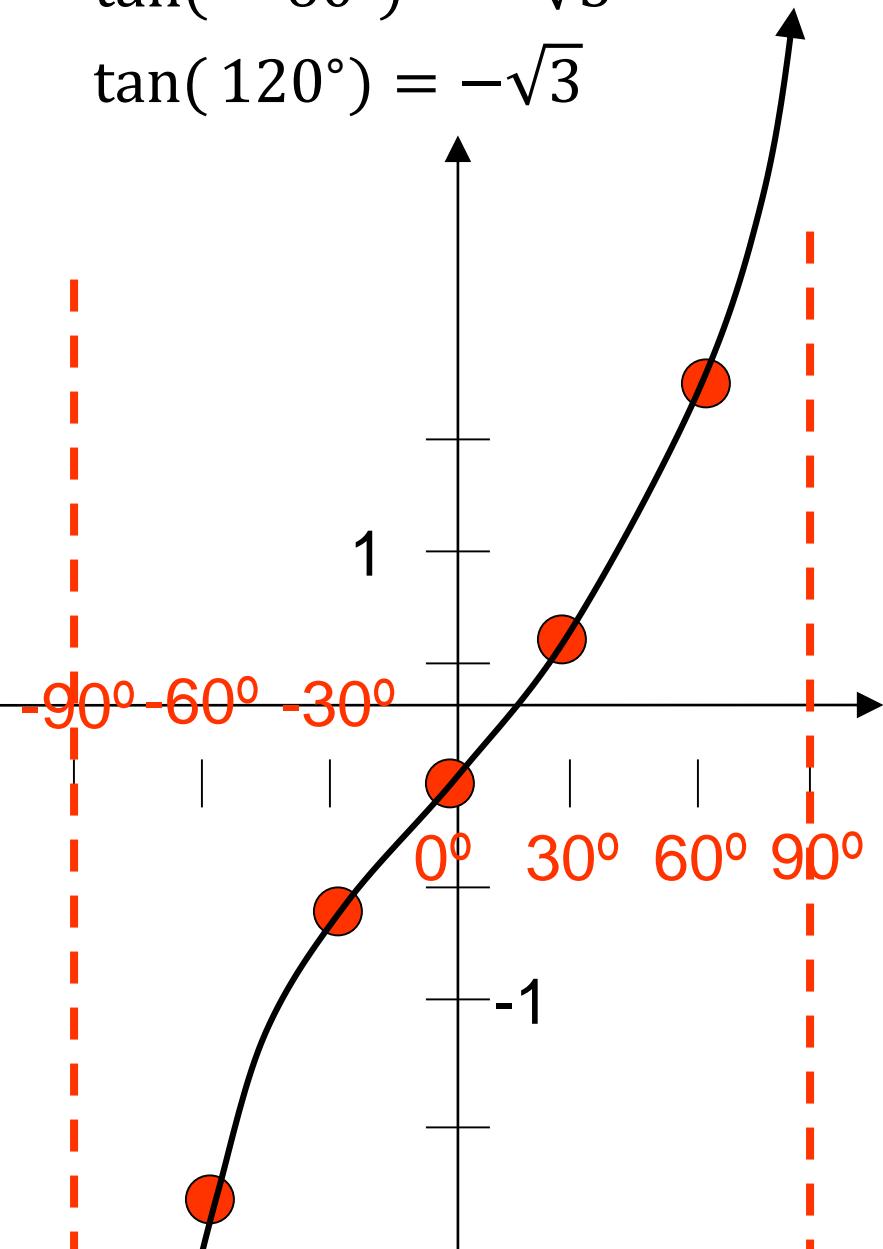
$$x = 60 + 360n \text{ or } 300 + 360n \text{ for } n \in \mathbb{Z}$$



Solve $\tan 2x = -\sqrt{3}$

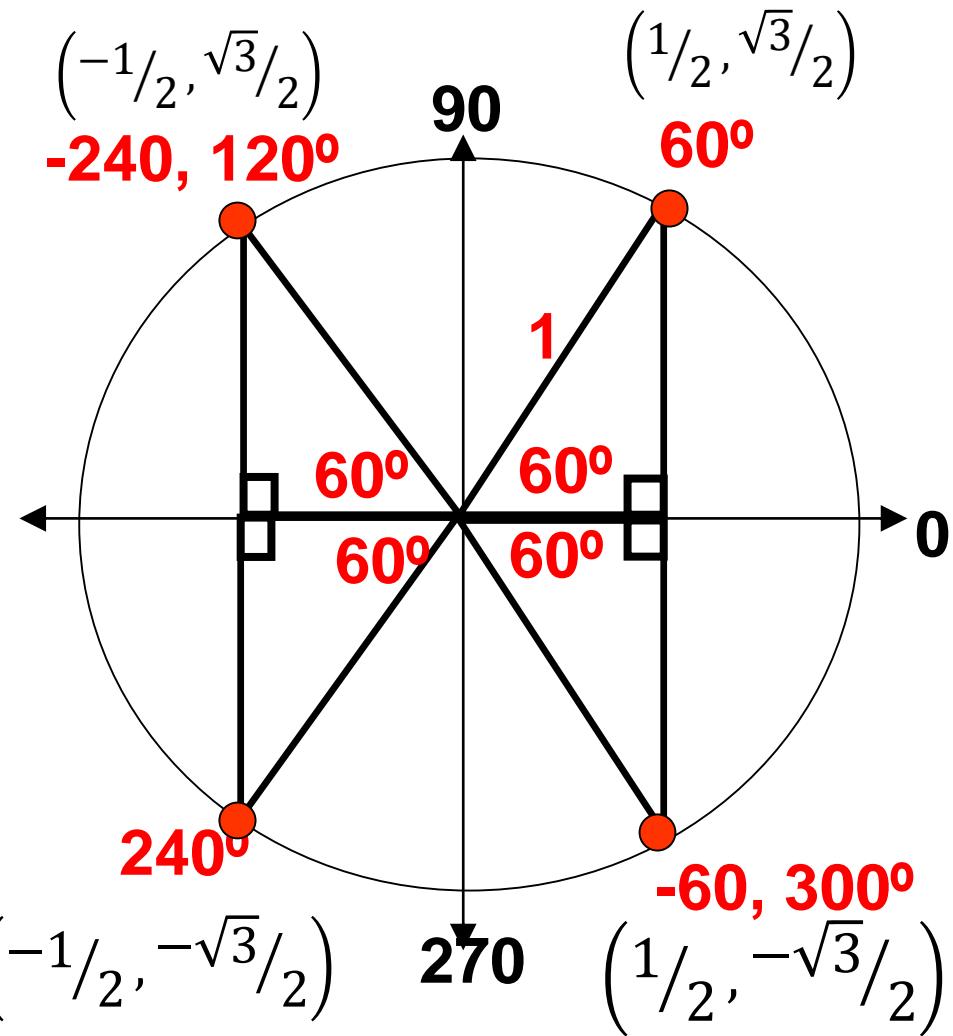
$$\tan(-60^\circ) = -\sqrt{3}$$

$$\tan(120^\circ) = -\sqrt{3}$$



$$x = 120 + 180n \text{ for } n \in \mathbb{Z}$$

$$x = \frac{2\pi}{3} + \pi n \text{ for } n \in \mathbb{Z}$$



Method 2: Solving using Algebra

Linear Trigonometric Equation (biggest exponent is a '1'):

$$4 \sin x - 2 = -4$$

To demonstrate that this is a linear equation, let $x = \sin x$

$$4x - 2 = -4$$

Quadratic Trigonometric Equation (biggest exponent is a '2'):

$$2\cos^2 x + \cos x - 1 = 0$$

To demonstrate that this is a quadratic equation, let $x = \cos x$

$$2x^2 + x - 1 = 0$$

Solve $4 \sin x - 2 = -4$ on the interval: $0 \leq x < 2\pi$

To solve; “isolate the sine, undo the sine”

$$4 \sin x = -2 \quad \text{Add '2' left/right}$$

$$\sin x = -\frac{1}{2} \quad \text{Divide by '4' left/right}$$

$$x = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

Solve $2\cos^2 x + \cos x - 1 = 0$ on the interval $0 \leq x < 2\pi$

Use substitution: let $x = \cos x$

$$2x^2 + x - 1 = 0$$

$$(2x - 1)(x + 1) = 0 \quad x = \frac{1}{2}, -1$$

Back-substitute: $\cos x = x$

$$\cos x = \frac{1}{2} \text{ or } \cos x = -1$$

$$x = \frac{\pi}{3}, \pi, \text{ or } \frac{5\pi}{3}$$

Solve $\tan^2 x - \tan x = 6$ on the interval $0 \leq x < 180$

Re-write as a standard form quadratic equation.

$$\tan^2 x - \tan x - 6 = 0$$

Use substitution or factor directly

$$(\tan x - 3)(\tan x + 2) = 0$$

$$\tan x - 3 = 0$$

$$\tan x + 2 = 0$$

$$\tan x = 3$$

$$\tan x = -2$$

x is not a “nice” angle.

x is not a “nice” angle.

$$\tan^{-1} (3) = x$$

$$\tan^{-1} (-2) = x$$

$$x = 71.6$$

$$x = 116.6$$

Solve $2\cos^2 \theta + 5\cos \theta - 6 = 0$ on the interval $0 \leq x < 360$

Use substitution: let $x = \cos \Theta$

$$2x^2 + 5x - 6 = 0$$

Try factoring: $2(-6) = -12$ $-12 = \underline{\hspace{2cm}} * \underline{\hspace{2cm}}$
 $5 = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$

There are no factors of -12 that add up to 5

Solve using the quadratic formula

$$x = \frac{-5}{2(2)} \pm \frac{\sqrt{(5)^2 - 4(2)(-6)}}{2(2)}$$

$$x = -1.25 \pm 2.136$$

$$x = -3.386 \text{ or } 0.8860$$

Back substitution: let $x = \cos \Theta$

$$\cos \theta = -3.386$$

Not possible

$$\cos \theta = 0.886$$

$$\cos^{-1} 0.886 = \theta$$

$$\theta = 27.6$$

Quadrant IV: $\theta = 360 - 27.6$

$$\theta = 332.4$$

Method 3: Solving using Trigonometric Identities

Solve $\cos x + \sin x = 1$ on the interval $0 \leq x < 2\pi$

Looks similar to the Pythagorean Identity

$$\cos^2 x + 2 \cos x \sin x + \sin^2 x = 1 \quad \text{Square left/right}$$

$$\cos^2 x + \sin^2 x + 2 \cos x \sin x = 1 \quad \text{Re-arrange}$$

$$1 + 2 \cos x \sin x = 1 \quad \text{Substitution (Pythag. ID)}$$

$$2 \cos x \sin x = 0 \quad \text{Subtract 1 left/right}$$

$$\cos x \sin x = 0 \quad \text{Divide 2 left/right}$$

$$\cos x = 0 \text{ or } \sin x = 0 \quad x = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or } \pi$$

$$\cos 0 + \sin 0 = 1 \quad \text{Check for extraneous solutions}$$

$$\cos \frac{\pi}{2} + \sin \frac{\pi}{2} = 1 \quad \cos \frac{3\pi}{2} + \sin \frac{3\pi}{2} \neq 1$$

$$\cos \pi + \sin \pi \neq 1$$

Common mistake: dividing to eliminate a variable (which variable we're trying to solve for) since the variable could equal zero → you can't divide by zero.

Solve $\sin 2x = \sin x$ on the interval $0 \leq x < 2\pi$

$$\sin 2x = 2\sin x \cos x \quad \text{Double angle identity}$$

Divide by $(\sin x)$ left/right is an error.

$$2\sin x \cos x = \sin x \quad \text{Substitute using Double Angle ID}$$

$$2\sin x \cos x - \sin x = 0 \quad \text{Subtract } (\sin x) \text{ left/right}$$

$$(\sin x)(2 \cos x - 1) = 0 \quad \text{Factor}$$

$$\sin x = 0 \text{ or } \cos x = \frac{1}{2}$$

$$x = 0, \pi, \frac{\pi}{3}, \text{ or } \frac{5\pi}{3}$$

Solve $\sin x + \csc x = -2$ Add 2 left/right

$$\sin x + \csc x + 2 = 0 \quad \text{Convert to sines/cosines}$$

$$\sin x + \frac{1}{\sin x} + 2 = 0 \quad \text{Multiply by } (\sin x) \text{ left/right}$$

$$\sin^2 x + 2 \sin x + 1 = 0$$

$$(\sin x + 1)(\sin x + 1) = 0 \quad \text{Solve}$$

$\sin x = -1$ Since no restricted domain given,
must solve for all real numbers.

$$x = 270 + 360n \text{ for } n \in \mathbb{Z}$$

$$x = \frac{3\pi}{2} + 2\pi n \text{ for } n \in \mathbb{Z}$$

Solve $3\cos^2 x + \sin x = 3$ on the interval $0 \leq x < 360$

$$\sin^2 x + \cos^2 x = 1$$

Convert cosines to sines
(Pythagorean ID)

$$3(1 - \sin^2 x) + \sin x = 3$$

Non-standard form
quadratic equation →
rewrite in standard form

$$3 - 3\sin^2 x + \sin x = 3$$

$$3\sin^2 x - \sin x = 0$$

Factor

$$\sin x(3\sin x - 1) = 0$$

Solve $\sin x = 0$ or $\sin x = \frac{1}{3}$

$$\sin x = 0$$

$$\sin x = \frac{1}{3} \quad \sin^{-1}\left(\frac{1}{3}\right) = x$$

$$x = 0 \text{ or } 180$$

$$x = 19.5$$

Quadrant II:

$$x = 160.5$$