Math-1060
Lesson 6-5

Inverse Trigonometric Functions

Relation: A "mapping" or pairing of input values to output values.

Function: A relation where each input has exactly one output.

There are at least 6 ways to show a relation between input and output values.

Data table:

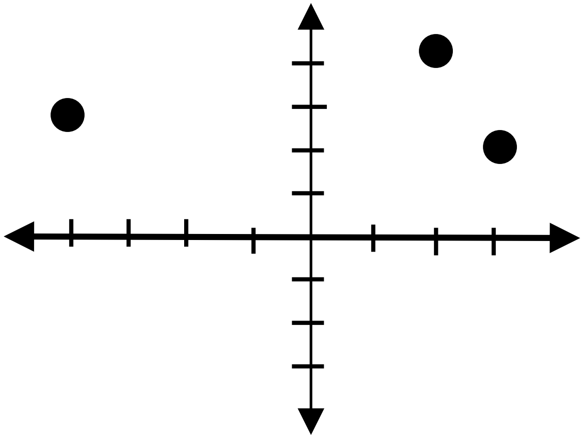
x	2	3	-4
y	4	2	3

Ordered Pairs:
(2, 4), (3, 2), (-4, 3)

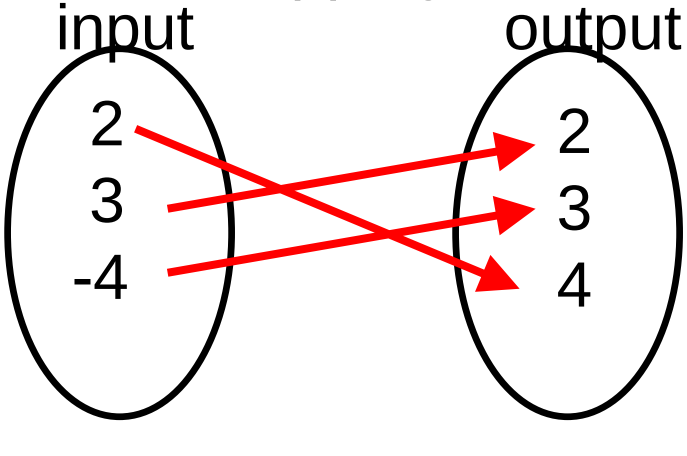
Equation: $y = 2x + 1$

Function notation: $f(2) = 4$

Graph:



Mapping



What is an "Inverse Function"?

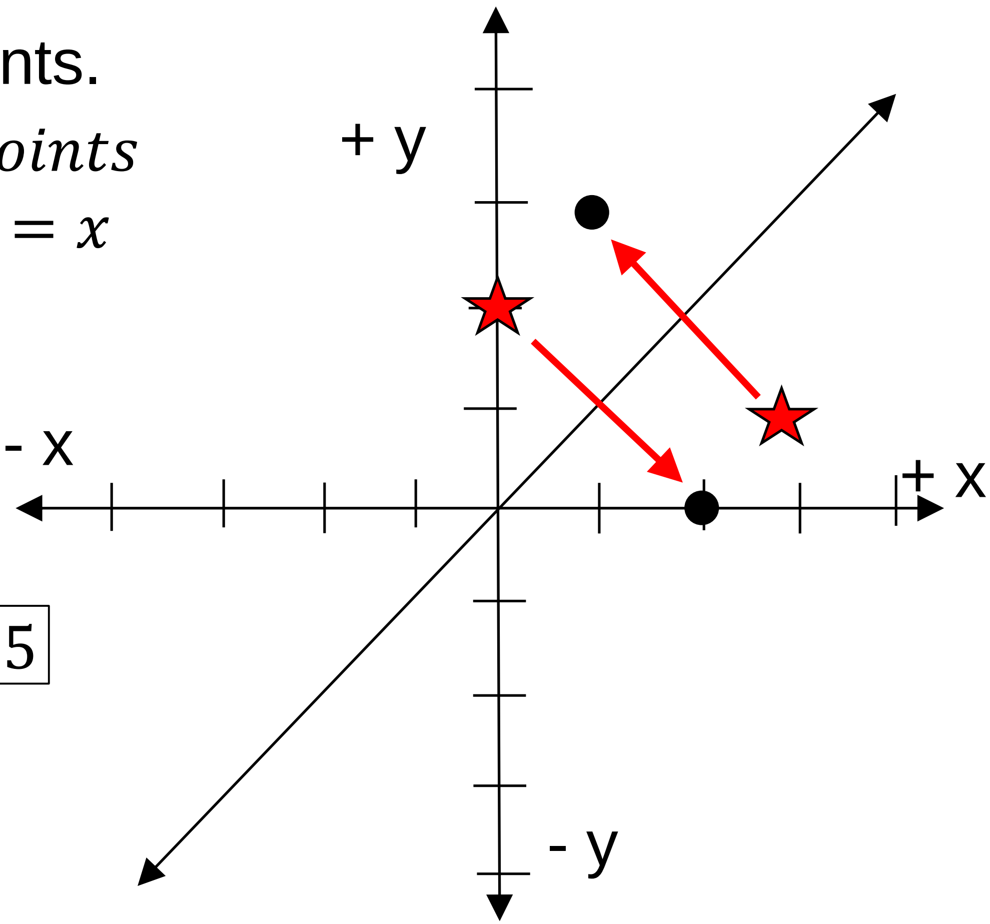
It depends upon which form of the function you're talking about.

Ordered Pairs: f(x) = (2, 4), (3, 2), (-4, 3)

$$f^{-1}(x) = (4, 2), (2, 3), (3, -4)$$

Graph: f(x) are the starred points.

$$f^{-1}(x)\ reflection\ of\ all\ points$$
$$in\ f(x)\ across\ the\ line\ y = x$$



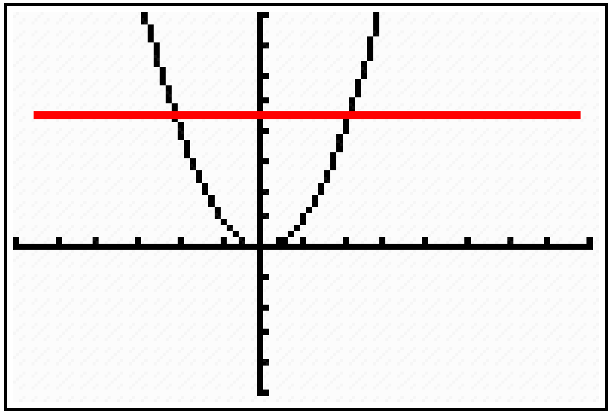
Equation: $\boxed{f(x) = x - 5}$

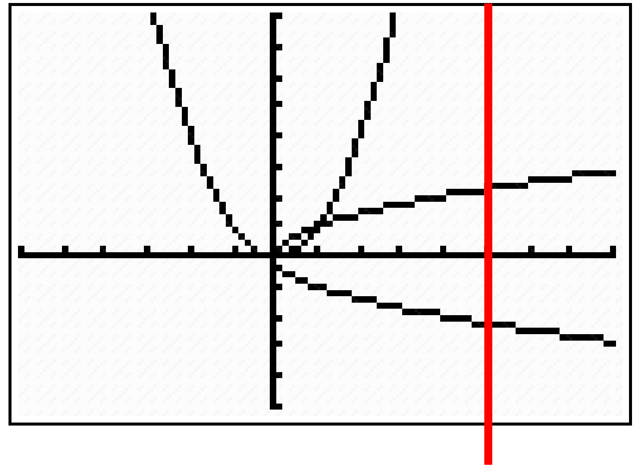
Exchange 'x' and 'y' in the equation then solve for 'y'

$$y = x - 5$$
$$x = y - 5$$
$$x + 5 = y$$

$\boxed{f^{-1}(x) = x + 5}$

Does reflecting the graph of the equation $f(x) = x^2$ across the line y = x result in a function?

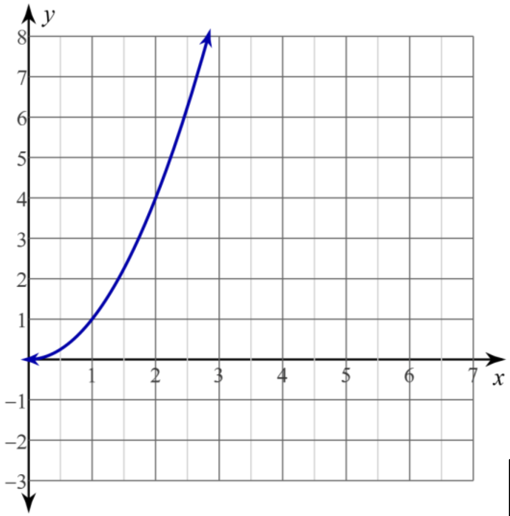




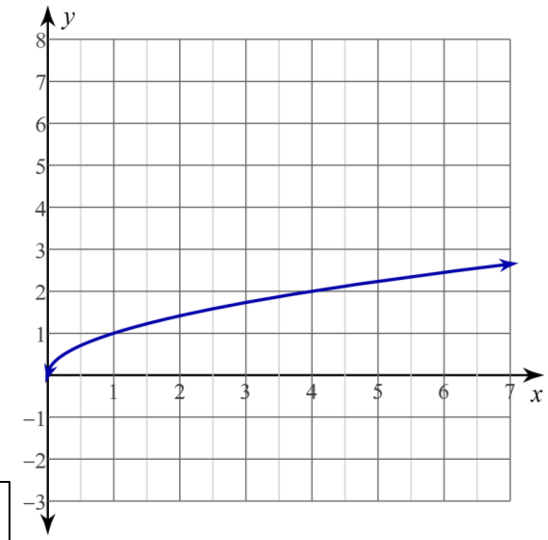
How can you tell by looking at the graph of a function if its inverse will be a function?

<u>Horizontal Line Test</u>: if a horizontal line intersects the graph more than once, then the inverse function will fail <u>*the vertical line test.*</u>

How can we say that $f(x) = x^2$ then $f^{-1}(x) = \sqrt{x}$?



If we only graph the right side of the parabola, then the square root is the inverse.



We "restrict" the domain.

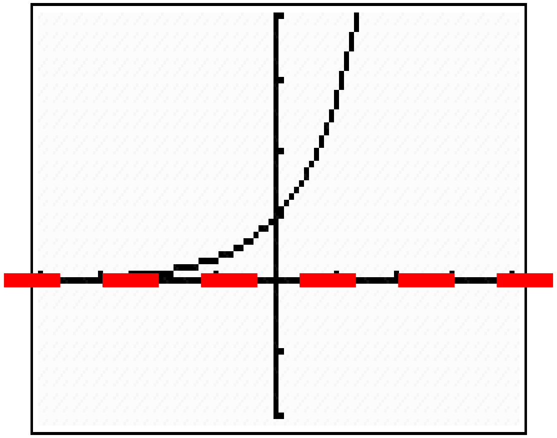
$f(x) = x^2$ for $x = [-1, \infty)$

$f^{-1}(x) = \sqrt{x}$

If you only want to graph part of the function, you limit the input values to those that will give you the part of the graph that you want graphed.

How does the domain/range of an original function relate to the domain/range of its inverse?

Exponential Function

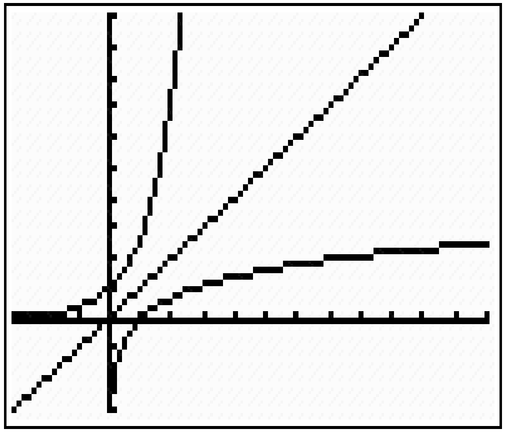


$f(x) = 10^x$

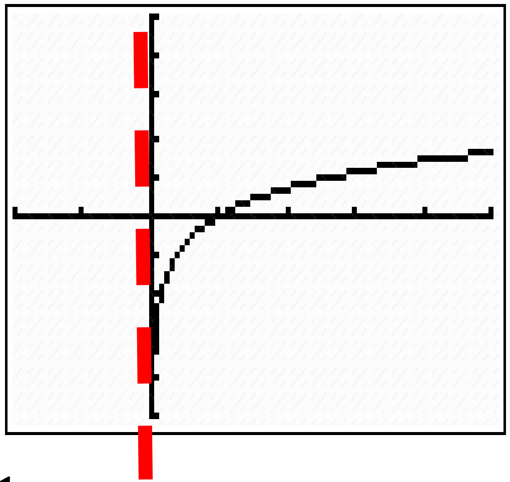
Domain = ? (-∞, ∞)

Range = ? (0, ∞)

(Inverse Functions)



Logarithm Function



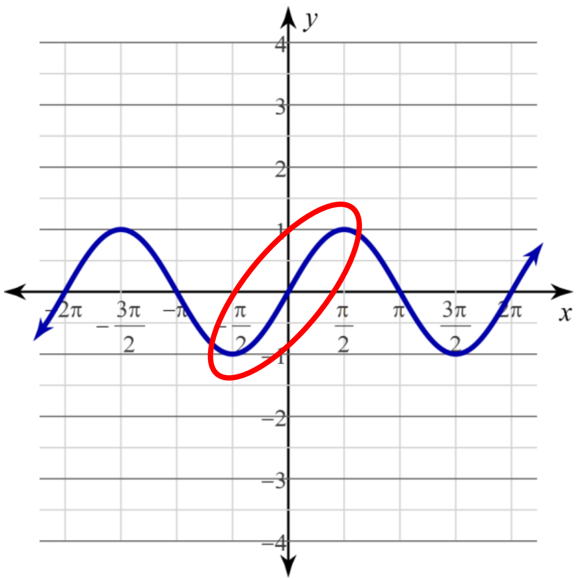
$f^{-1}(x) = log_{10}(x)$

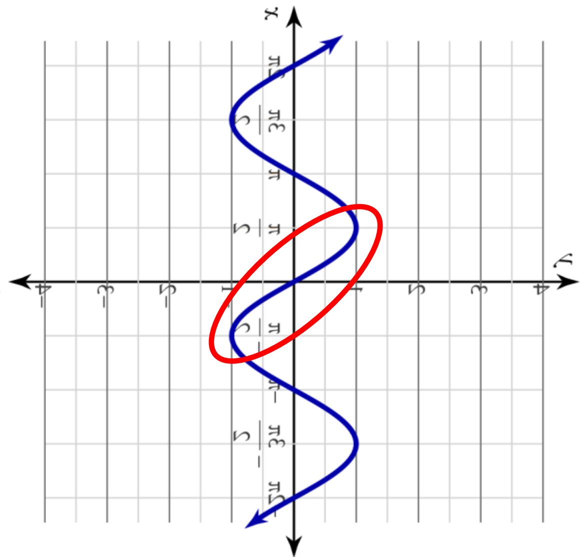
Domain = ? (0, ∞)

Range = ? (-∞, ∞)

Exchange ‘x’ and ‘y’ in the equation then solve for ‘y’ → domain and range are exchanged.

$f(x) = \sin x$ $\qquad f^{-1}(x) = \sin^{-1}(x)$





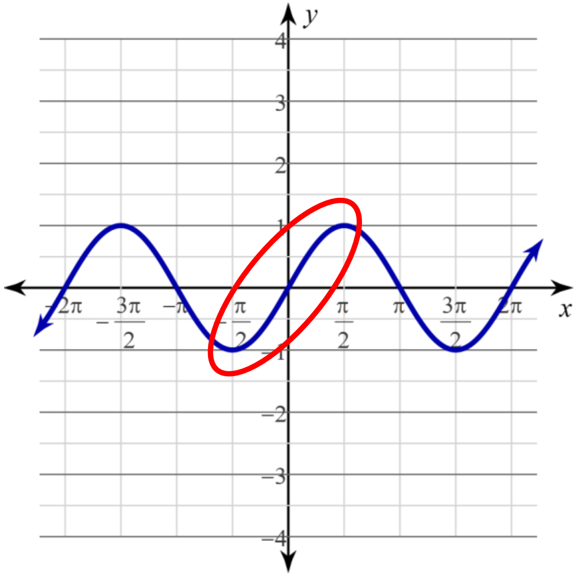
How do we "restrict" the domain of the <u>sine function</u> so that the Inverse Sine function (arcsine function) will be a function?

If we reflect the sine function across the line y = x, will the resulting relation be a function?

NO

We restrict the domain so that only the portion of the graph that passes the horizontal line test will be graphed.

$f(x) = \sin x$

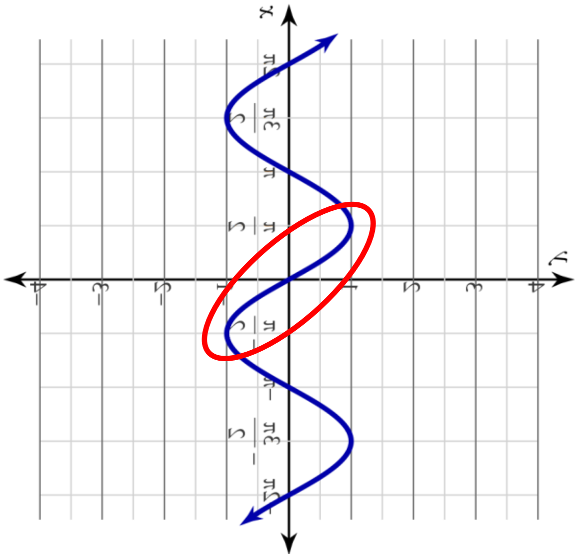


Domain = ? $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Range = ? [-1, 1]

What is the *restricted domain*, and the *range* of the sine function?

$f^{-1}(x) = \sin^{-1}(x)$



Domain = ? [-1, 1]

Range = ? $[-\frac{\pi}{2}, \frac{\pi}{2}]$

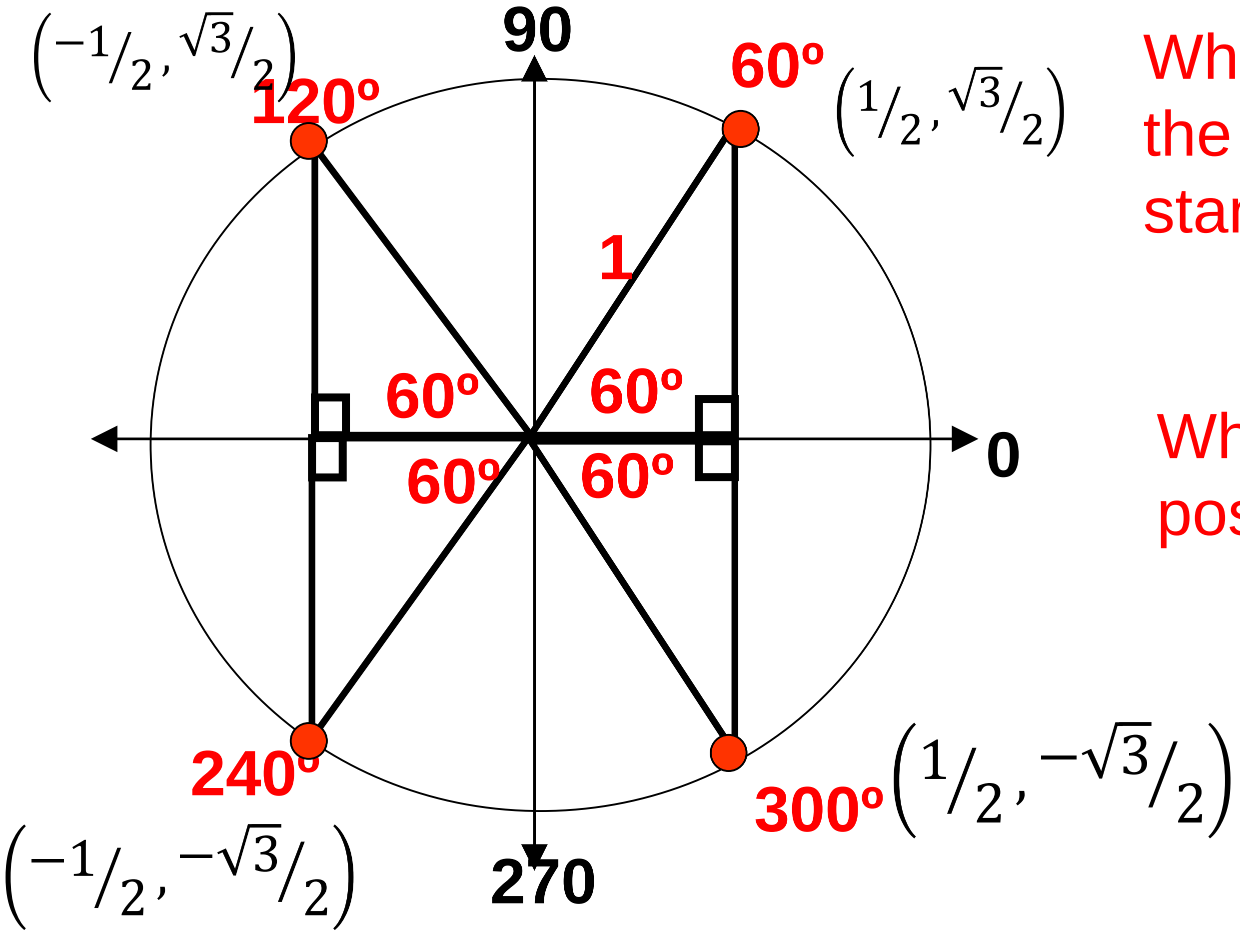
What is the domain and range of the arcsine function?

Review

Find the exact value of θ without using a calculator (in degrees).

$$\sin\theta = -\sqrt{3}/2$$

What is the reference angle? 60º



What is the quadrant of the terminal side of the standard position angle?

III or IV

What is the standard position angle?

240º or 300º

Composition of Trigonometric functions:

$$\left(\sqrt{x}\right)^2 = x$$

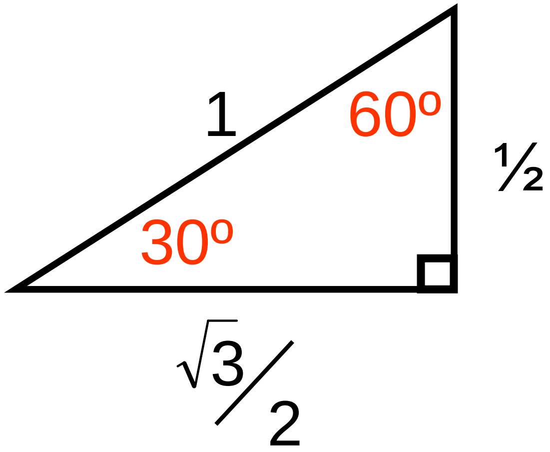
Think of a function and its inverse function of "<u>undoing</u>" each other.

$$\sin\left(\sin^{-1}(x)\right) = x$$

$$\cos^{-1}\left(\cos(x)\right) = x$$

Whenever you compose a function and its inverse, they "cancel" or "undo" each other, leaving just the variable.

Inverse Sine Function (Arcsine Function)



$$\sin 30° = {}^{1}/_{2}$$

Sine of an angle = ratio

$$\sin \theta = {}^{o}/_{h}$$

$$\sin^{-1}\left({}^{1}/_{2}\right) = 30°$$

Arcsine of a ratio = angle

$$\sin^{-1}\left({}^{o}/_{h}\right) = \theta$$

Sine-Inverse Sine Identities

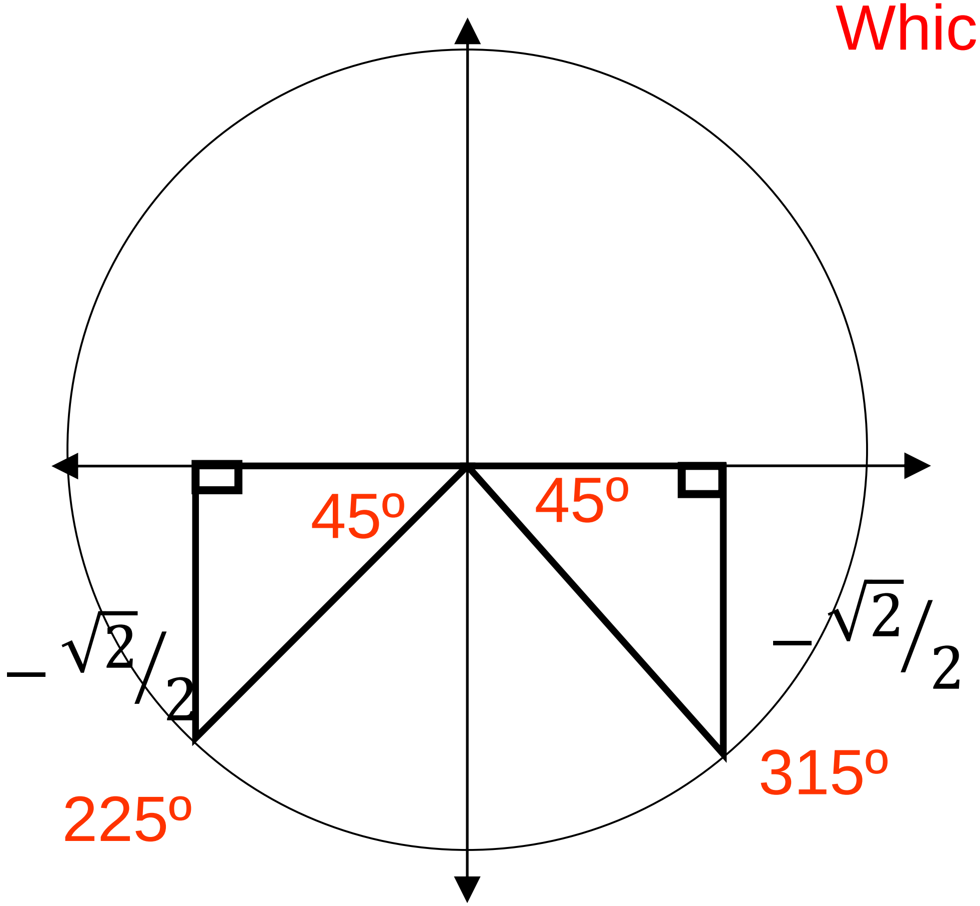
$$\sin(\sin^{-1} x) = x \quad for\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\sin^{-1}(sin\, x) = x$$

$$\sin(\sin^{-1}(\frac{-3\pi}{5}) = \frac{-3\pi}{5}$$

1. Find the <u>exact value</u> (use unit circle) of (in degrees): $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Which of the two angles is it?



Remember, for the arcsine function, we restrict the domain of the sine function to the interval: $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$$-90 \le x \le 90$$

Answer: 315

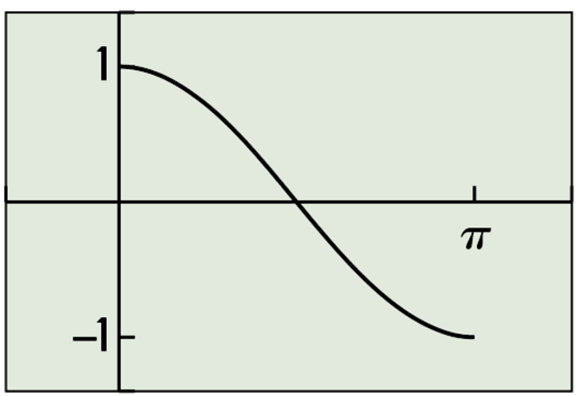
Inverse Cosine Function

$$\cos(\theta) = \frac{adj}{hyp}$$ Cosine of an angle = ratio

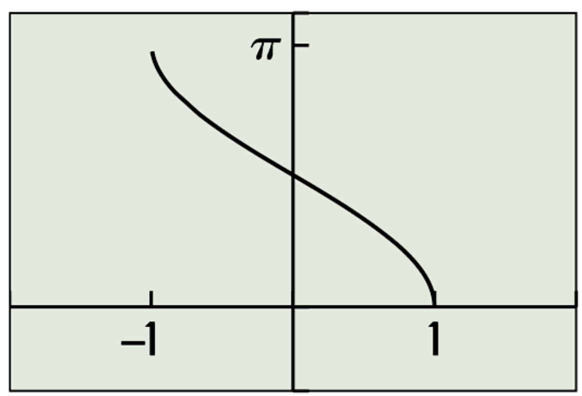
$$\cos^{-1}\left(\frac{adj}{opp}\right) = \theta$$ Arccosine of a ratio = angle

On the interval $0 \le x \le \pi$ or 0 ≤ x ≤ 90, cos(x) and its inverse are one-to-one functions (every output has exactly one input) and every input has exactly one output.

Notice the different domains/ranges



[–1, 4] by [–1.4, 1.4]
(a)

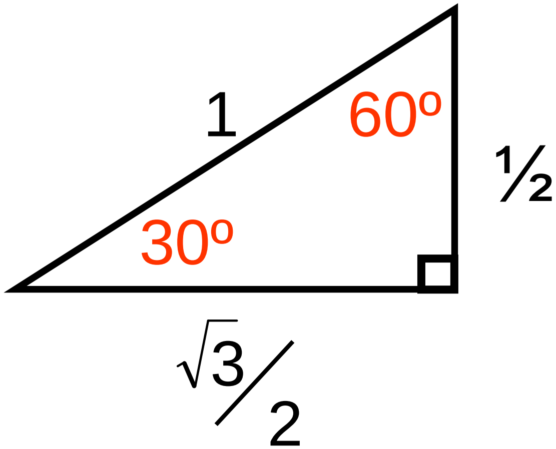


[–2, 2] by [–1, 3.5]
(b)

Inverse Cosine (Arccosine Function)

The unique angle y in the interval $[0,\pi]$ such that $\cos y = x$ is the **inverse cosine** (or **arccosine**) of x, denoted $\mathbf{cos^{-1}}\,\boldsymbol{x}$ or **arccos** $\boldsymbol{x}$.

The domain of $y = \cos^{-1} x$ is [-1,1] and the range is $[0,\pi]$.



$$\cos 30^\circ = \sqrt{3}\big/2$$

$$\cos^{-1}\left(\sqrt{3}\big/2\right) = 30^\circ$$

Cosine-Inverse Cosine Identities

$$\cos(\cos^{-1} x) = x \qquad for\; 0 \le x \le \pi$$

$$\cos^{-1}(cos\,x) = x$$

$$\cos(cos^{-1}\frac{3\pi}{5}) = \frac{3\pi}{5}$$

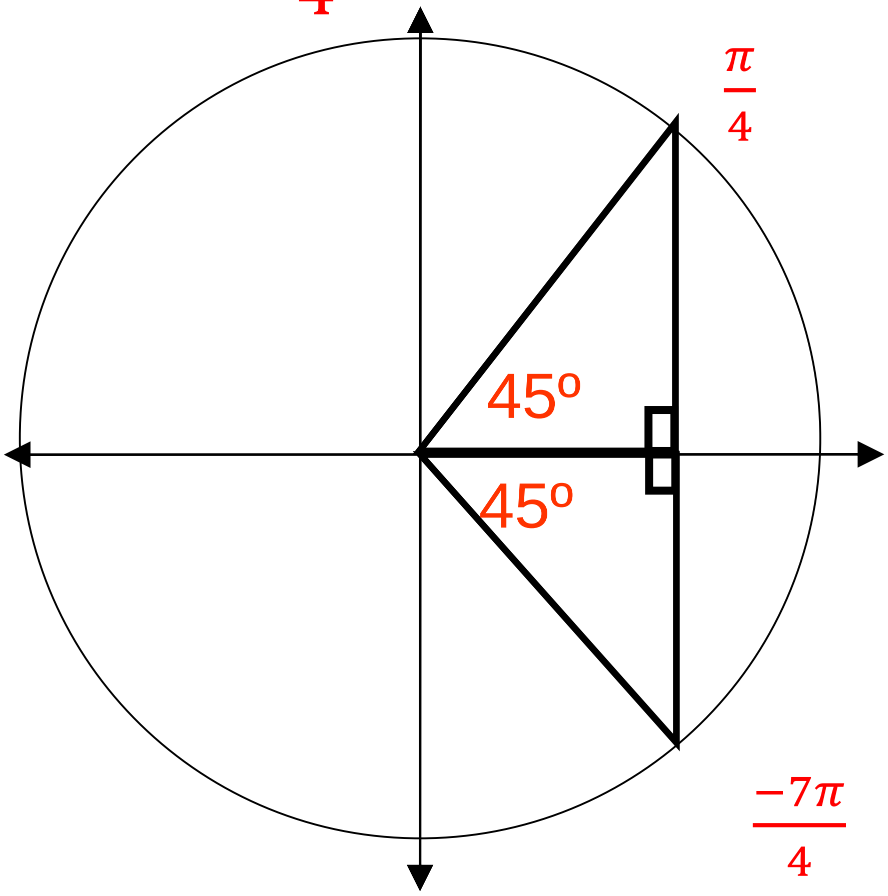
Cosine-Inverse Cosine Identities

$$\cos(\cos^{-1} x) = x \qquad for\ 0 \le x \le \pi$$

$$\cos^{-1}(\cos x) = x$$

$$\cos^{-1}\left(cos\frac{-7\pi}{4}\right) =?$$

$\frac{-7\pi}{4}$ is not in the domain of the arc-cos function.



$$\cos^{-1}\left(cos\frac{\pi}{4}\right) =? \qquad \sqrt{2}/_2$$

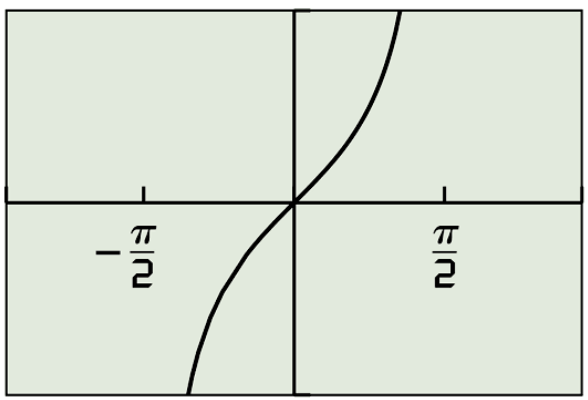
Inverse Tangent Function (Arctangent Function)

$$\tan(\theta) = \frac{opp}{adj}$$ tangent of an <u>angle</u> = <u>ratio</u>

$$\tan^{-1}\left(\frac{opp}{adj}\right) = \theta$$ Arctangent of a <u>ratio</u> = <u>angle</u>

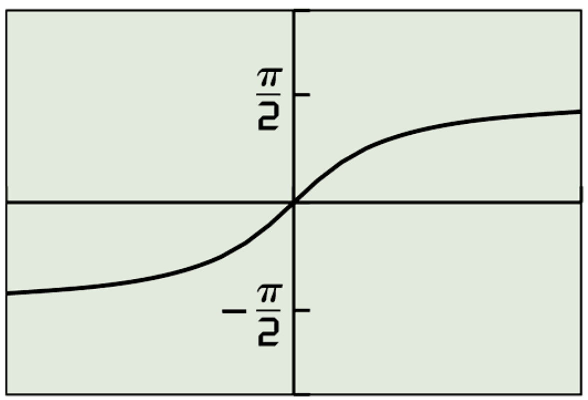
On the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ or -90 ≤ x ≤ 90, tan(x) and its inverse are one-to-one functions.

Tangent



[–3, 3] by [–2, 2]
(a)

Arctangent

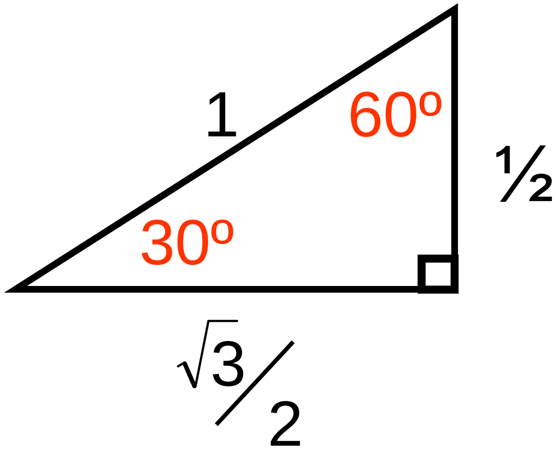


[–4, 4] by [–2.8, 2.8]
(b)

Inverse Tangent Function (Arctangent Function)

The unique angle y in the interval $(-\pi/2, \pi/2)$ such that $\tan y = x$ is the **inverse tangent** (or **arctangent**) of x, denoted $\mathbf{tan}^{-1}\,\boldsymbol{x}$ or $\mathbf{arctan}\,\boldsymbol{x}$.

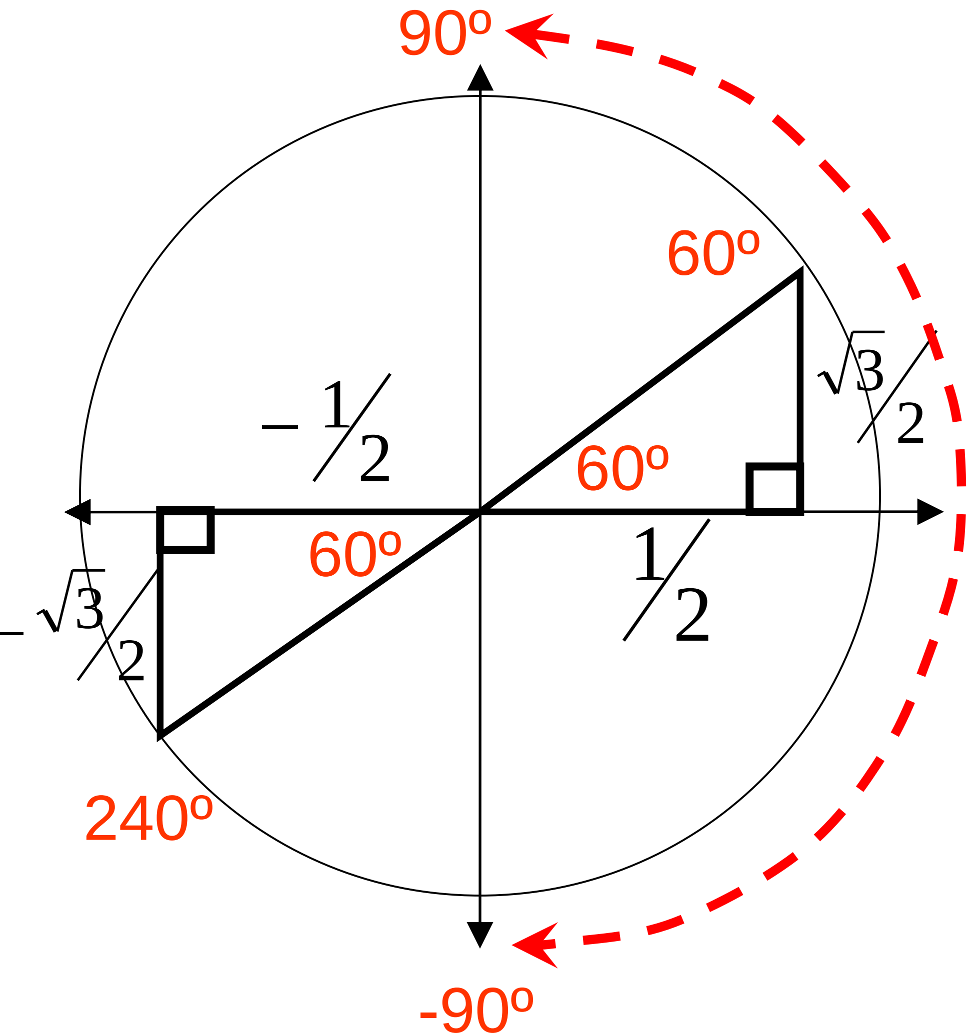
The domain of $y = \tan^{-1} x$ is $(-\infty, \infty)$ and the range is $(-\pi/2, \pi/2)$.



$$\tan 30^\circ = \sqrt{3}/3$$

$$\tan^{-1}\left(\sqrt{3}/3\right) = 30^\circ$$

Find the exact value (use unit circle) of $\tan^{-1}\sqrt{3}$ (in degrees):



$$\frac{y}{x} = \sqrt{3} = \frac{-\sqrt{3}/2}{-1/2} = \frac{\sqrt{3}/2}{1/2}$$

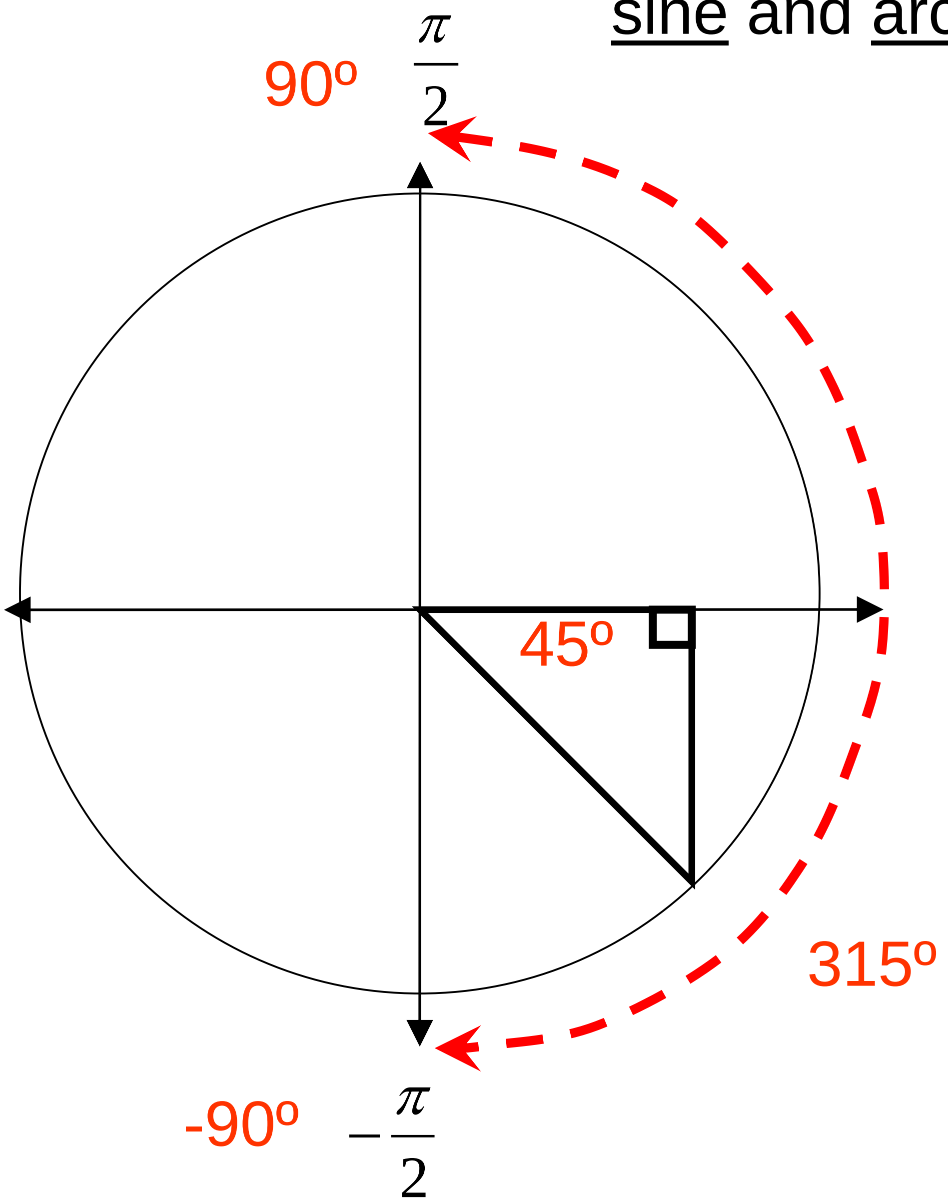
Which angle is it? (60 or 240)

Remember, we use tan 'x' on the interval:

$$-90 \leq x \leq 90$$

Answer: 60

On what interval (of the circle) are the sine and arcsine functions one-to-one?



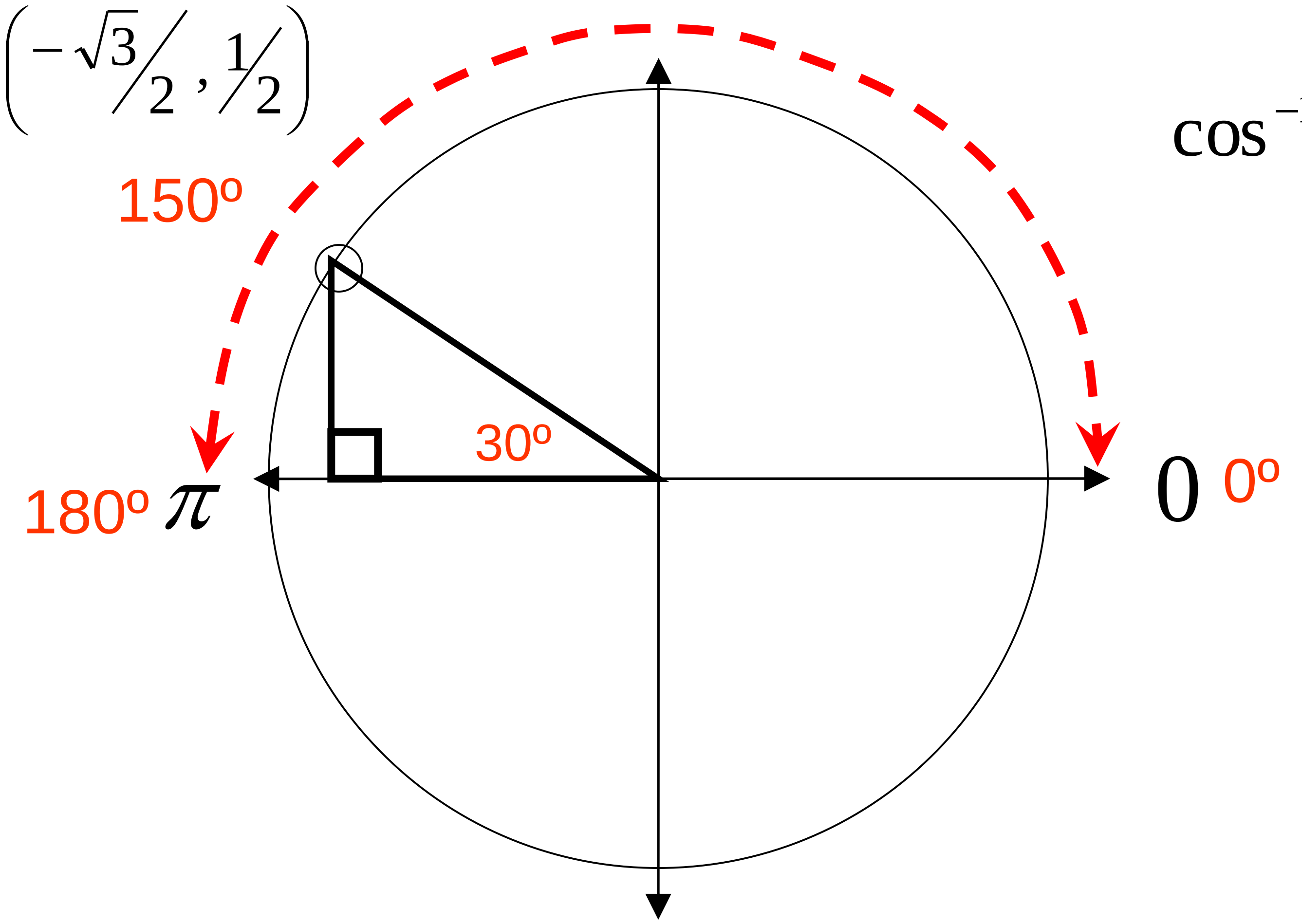
$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \theta$$

$$\theta = ?$$

Answer: 315

On what interval are the cosine and arc-cosine functions one-to-one? $0 \le \theta \le \pi$

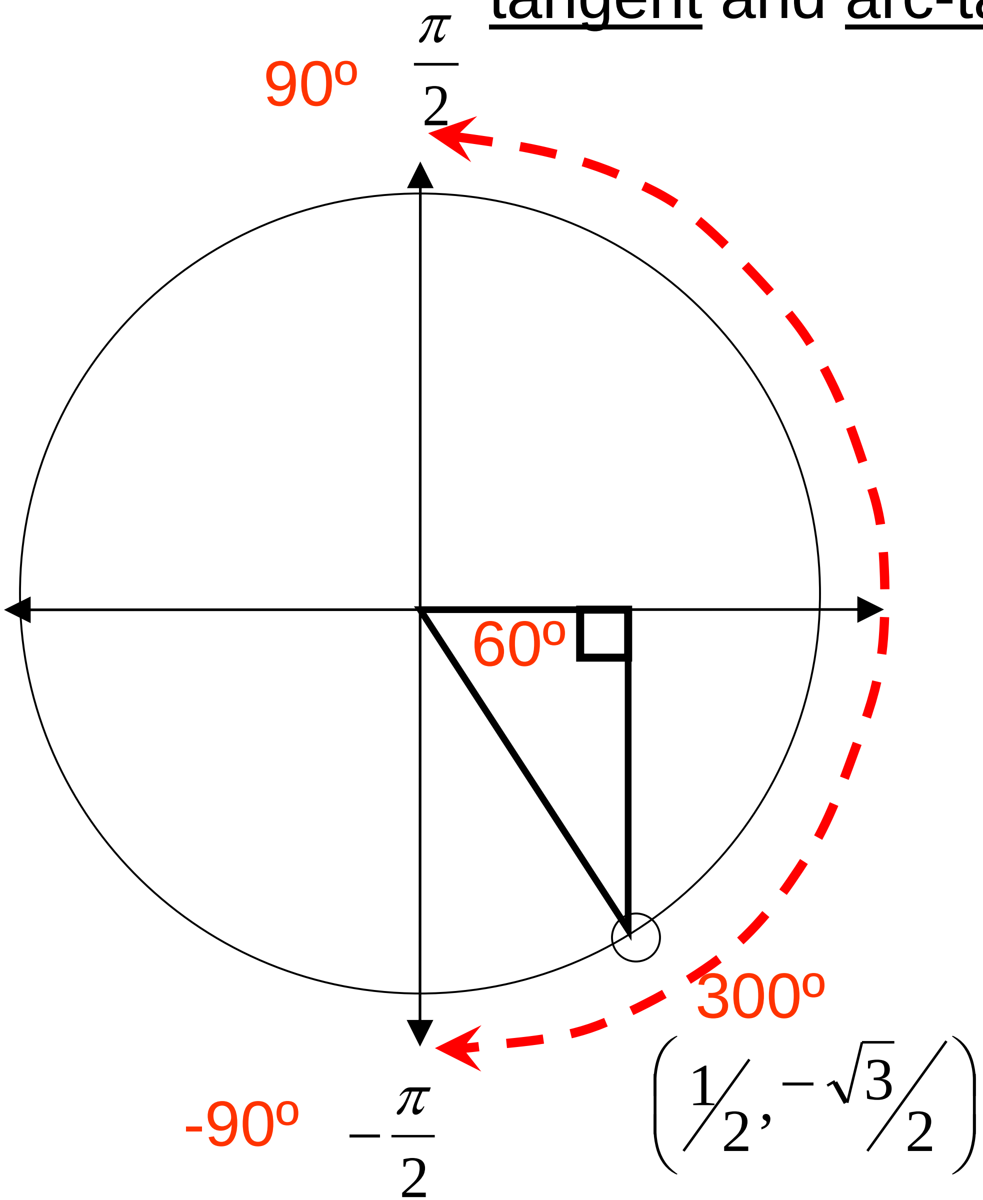


$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta$$

$$\theta = ?$$

$$\boxed{\theta = 150^\circ}$$

On what interval (of the circle) are the <u>tangent</u> and <u>arc-tangent</u> functions one-to-one?



$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\tan^{-1}\left(-\sqrt{3}\right) = \theta$$

$$\theta = ?$$

$$\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

$$\theta = 300^{\circ}$$

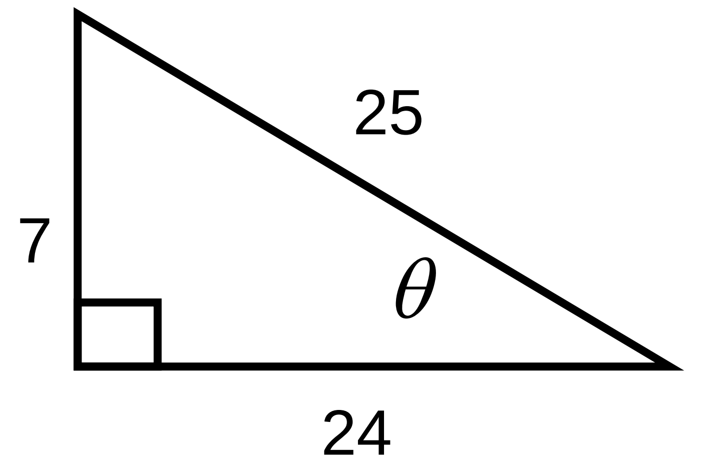
Using inverse trig. Functions to solve triangles.

Use the sine ratio to find θ

$$\sin\theta = \left(\frac{7}{25}\right)$$

$$\sin^{-1}(\sin\theta) = \sin^{-1}\left(\frac{7}{25}\right)$$

$$\theta = \sin^{-1}\left(\frac{7}{25}\right) \qquad \theta = 16.3^\circ$$



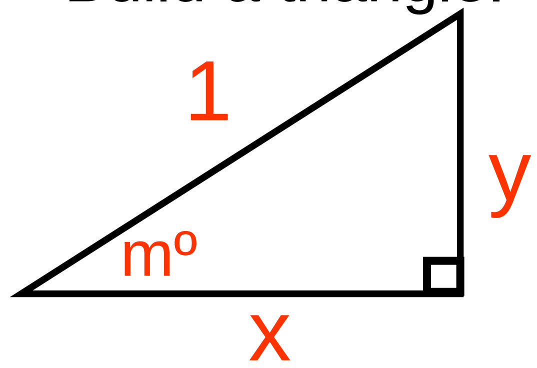
$$\cos\left(\sin^{-1}(y)\right) = ?$$

Trig functions composed with other trig functions that are <u>not inverses of each other</u>.

What is the angle?

$$\sin^{-1}\left(\frac{y}{1}\right) = ? \quad = m^\circ$$

Build a triangle.



By substitution:

$$\cos\left(\sin^{-1}(y)\right) = \cos(m^\circ)$$

What is the cosine ratio for angle m? $\sqrt{1-y^2}$

$$\left(\frac{x}{1}\right)$$

We need the ratio in terms of ‘y’.

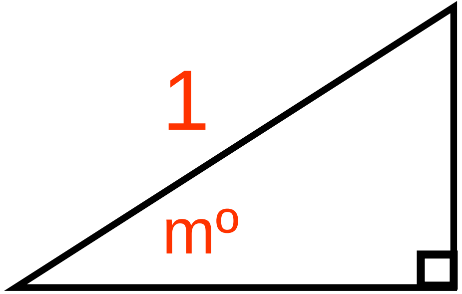
$$x^2 + y^2 = 1^2$$

$$x = \pm\sqrt{1-y^2}$$

$$\cos\left(\sin^{-1}(y)\right) = \cos(m^\circ)$$

$$\cos(m) = \sqrt{1-y^2}$$

$\tan\left(\cos^{-1} x\right) = ?$



$y = \sqrt{1 - x^2}$

$x = \sqrt{1 - y^2}$

What angle has a cosine ratio of x? $\cos^{-1} x = ? \quad = m^\circ$

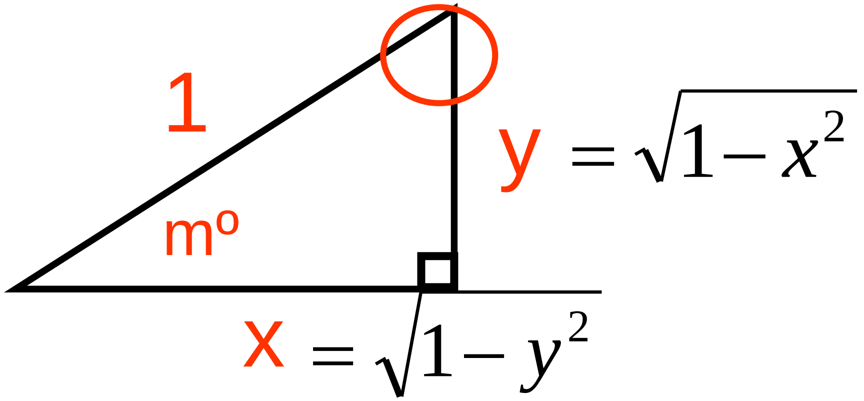
By replacement: $\tan(\cos^{-1} x) = \tan\left(m^\circ\right)$

What are the tangent ratios of for angle m?

$$= \frac{y}{x} \quad = \frac{y}{\sqrt{1 - y^2}} \quad = \frac{\sqrt{1 - x^2}}{\sqrt{1 - y^2}} = \frac{\sqrt{1 - x^2}}{x}$$

→ Answer should only have ‘x’ in it.

$$\csc(\arccos y) = ?$$



→ Answer should only have 'y' in it.

$$\arccos y = \cos^{-1} y/1$$

Which angle has a cosine ratio of y/1 ?

What is the measure of the angle that has a cosine ratio of y/1 ?

$$\angle = (90 - m^\circ) \qquad \cos^{-1} y = (90 - m^\circ)$$

$$\csc(\arccos y) = ? \quad = \csc(90 - m^\circ) \quad = \frac{hyp}{opp} \qquad = \frac{1}{\sqrt{1-y^2}}$$