

Math-1060
Lesson 6-4
“Product-to-Sum” Identity
and
“Sum-to-Product” Identity

The Product-to-Sum Identities come from the Sum and Difference Identities

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$$

$$\frac{1}{2}(\cos(A + B) + \cos(A - B)) = \cos A \cos B$$

Add the Cosine sum and Cosine Difference Identities together

What property of mathematics allows you to add equations together?

The Property of Equality

Divide by 2

Cosine Product-to-Sum Identity

$$\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B))$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Subtract the Cosine Sum Identity
from the Cosine Difference Identity

$$-(\cos(A + B)) = -(\cos A \cos B - \sin A \sin B)$$

$$\cos(A - B) - \cos(A + B) = 2 \sin A \sin B \quad \text{Divide by 2}$$

$$\frac{1}{2}(\cos(A - B) - \cos(A + B)) = \sin A \sin B$$

Sine Product-to-Sum Identity

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Add the Sine sum and Sine Difference Identities together

$$\sin(A + B) + \sin(A - B) = 2\sin A \cos B \quad \text{Divide by 2}$$

$$\frac{1}{2}(\sin(A + B) + \sin(A - B)) = \sin A \cos B$$

Sine Cosine Product-to-Sum Identity

$$\sin A \cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B))$$

Cosine Product-to-Sum Identity

$$\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B))$$

Sine Product-to-Sum Identity

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$

Sine Cosine Product-to-Sum Identity

$$\sin A \cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B))$$

Sine Product-to-Sum Identities

$$\sin A \sin B = \frac{1}{2} (\cos(A - B) - \cos(A + B))$$

Show that the identity is
true for: $A = 60^\circ$, $B = 30^\circ$

$$\sin 60^\circ \sin 30^\circ = \frac{1}{2} (\cos(60^\circ - 30^\circ) - \cos(60^\circ + 30^\circ))$$

$$\frac{\sqrt{3}}{2} * \frac{1}{2} = \frac{1}{2} (\cos(30^\circ) - \cos(90^\circ))$$

$$\frac{\sqrt{3}}{2} * \frac{1}{2} = \frac{1}{2} \left(\frac{\sqrt{3}}{2} + 0 \right)$$

$$\frac{\sqrt{3}}{2} * \frac{1}{2} = \frac{1}{2} * \frac{\sqrt{3}}{2}$$

true

Convert a Product to a sum

Let: $A = 5x$ and $B = 3x$

$$\sin A \sin B = \frac{1}{2} (\cos(A - B) - \cos(A + B))$$

$$\sin(5x)\sin(3x) = \frac{1}{2}(\cos(5x - 3x) - \cos(5x + 3x)) = \frac{1}{2}(\cos(2x) - \cos(8x))$$

$$\sin(5x)\sin(3x) = \frac{1}{2}(\cos(2x) - \cos(8x))$$

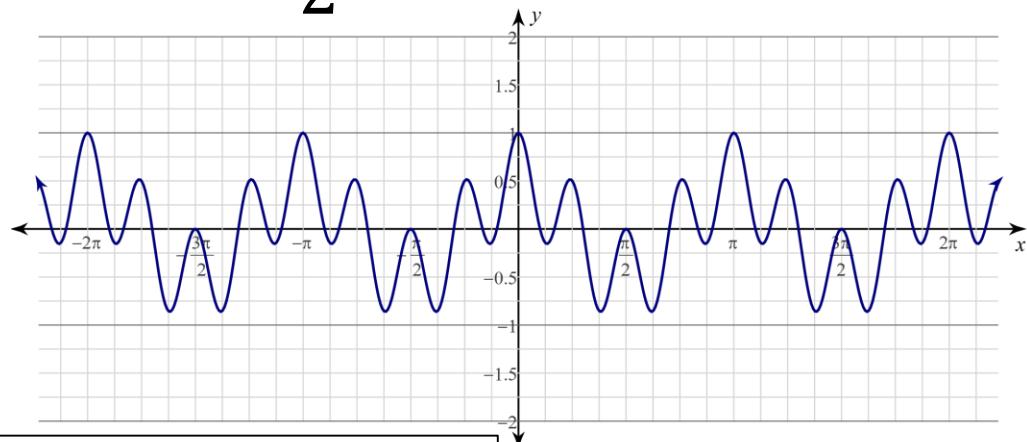
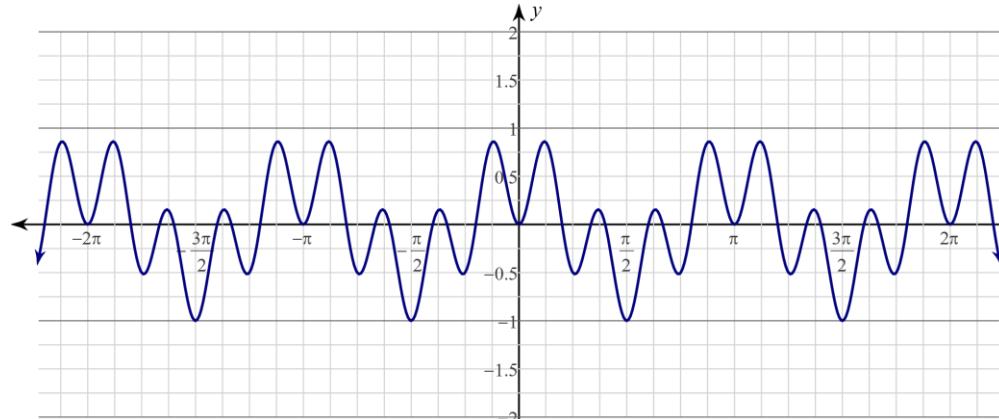
$$\cos A \cos B = \frac{1}{2} (\cos(A + B) + \cos(A - B))$$

$$\cos(5x)\cos(3x) = \frac{1}{2}(\cos(5x + 3x) + \cos(5x - 3x)) = \frac{1}{2}(\cos(8x) + \cos(2x))$$

$$\cos(5x)\cos(3x) = \frac{1}{2}(\cos(8x) + \cos(2x))$$

Set: x-min: -360
x-max: 360
y-min: -3
y-max: 3

Graph: $y_1 = \sin(5x)\sin(3x)$ Graph: $y_2 = \frac{1}{2}(\cos(8x) + \cos(2x))$

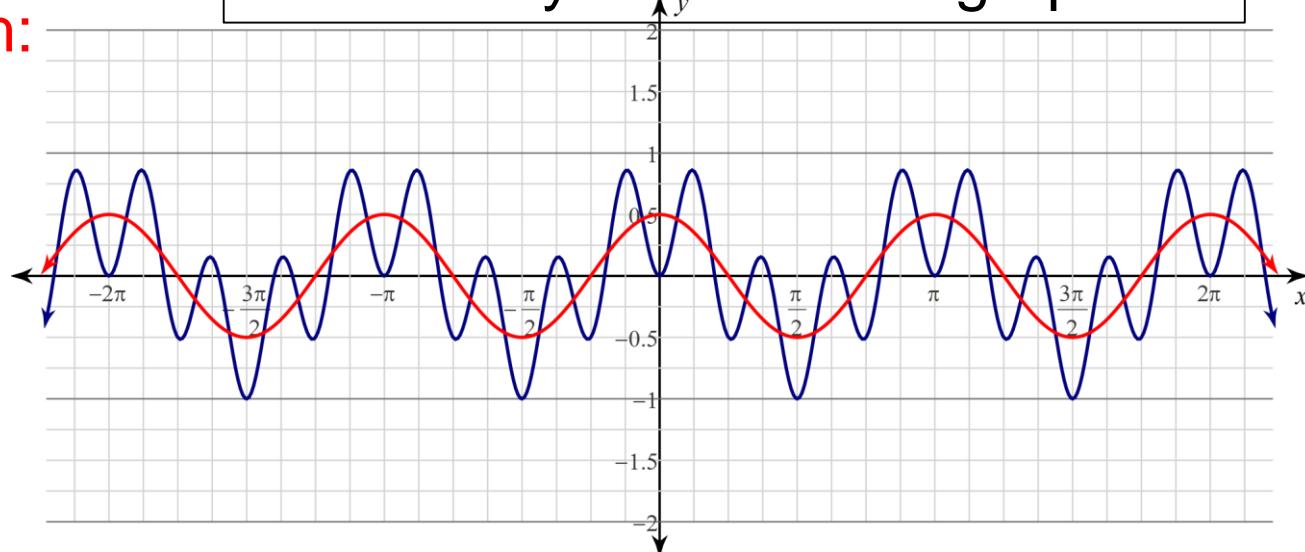


The identity is true when graphed.

At the same time graph:

$$y_1 = \sin(5x)\sin(3x)$$

$$y_2 = 0.5\cos(2x)$$



Instead of having a constant value for the amplitude, the amplitude is varying from -1 to +1 with a period of 72° .

Sine Sum-to-Product Identity Start with Sine Product-to-Sum Identity

$$\sin x \sin y = \frac{1}{2} (\sin(x+y) + \sin(x-y))$$

Let: $x+y = A$ Add the equations
And $x-y = B$ then solve for 'x'

Then $x = \frac{A+B}{2}$

Subtract the equations
then solve for 'y'

Then $y = \frac{A-B}{2}$

Then $x+y = \frac{A+B}{2} + \frac{A-B}{2} = A$

Then $x-y = \frac{A+B}{2} - \frac{A-B}{2} = B$

$$\sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) = \frac{1}{2} (\sin(A) + \sin(B))$$

Substitution into the Identity

Sine Sum-to-Product Identity

$$\sin(A) + \sin(B) = 2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

Multiply by 2 (and
switch sides)

We can use the same method to find the other “Sum-to-Product” Identities.

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin(A) - \sin(B) = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Show that

$$\sin(A) - \sin(B) = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

is true for:

A = 60 and

B = 30

$$\sin(60) - \sin(30) = 2\sin\left(\frac{60-30}{2}\right)\cos\left(\frac{60+30}{2}\right)$$

$$\frac{\sqrt{3}}{2} - \frac{1}{2} = 2\sin(15)\cos(45) = 2\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) * \frac{\sqrt{2}}{2}$$

(From lesson 6-3: $\sin(15) = \sin(60 - 45)$) $= \frac{\sqrt{6}-\sqrt{2}}{4}$

$$\frac{\sqrt{3}}{2} - \frac{1}{2} = 2\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) * \frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{3}-1}{2} = \frac{\sqrt{6}\sqrt{2} - \sqrt{2}\sqrt{2}}{4} = \frac{2\sqrt{3}-2}{4} = \frac{2(\sqrt{3}-1)}{2*2}$$

Q.E.D.

Convert $-4[\sin(6x) + \sin(12x)]$ into a product

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\begin{aligned}-4[\sin(6x) + \sin(12x)] &= -4[2\sin\left(\frac{6x+12x}{2}\right)\sin\left(\frac{6x-12x}{2}\right)] \\&= -8\sin(9x)\sin(-3x) \\&= 8\sin(9x)\sin(3x)\end{aligned}$$

Simplifying trigonometric expressions

$$4\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) - 6\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$2[2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)] - 3[2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)]$$

$$[\sin(x) + \sin(y)]$$

$$[\sin(x) - \sin(y)]$$

$$2[\sin(x) + \sin(y)] - 3[\sin(x) - \sin(y)]$$

$$2\sin(x) - 3\sin(x) + 2\sin(y) - 3\sin(y)$$

$$-\sin(x) - \sin(y)$$

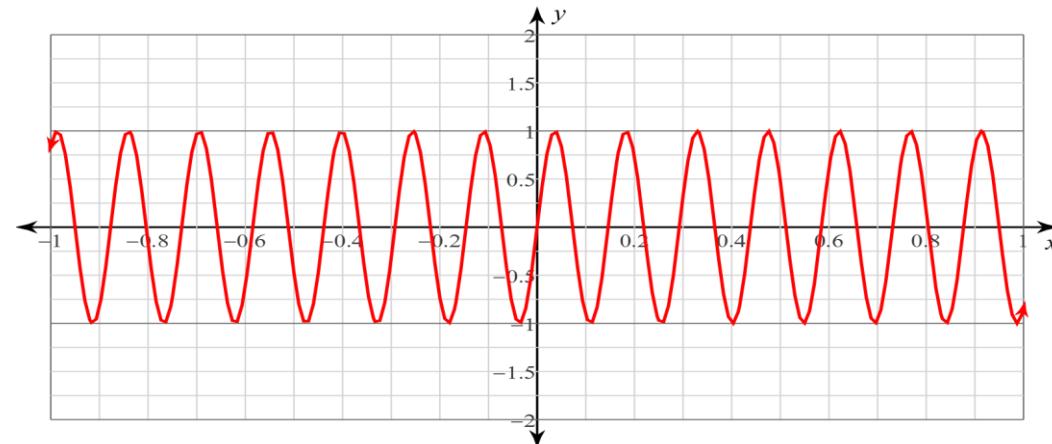
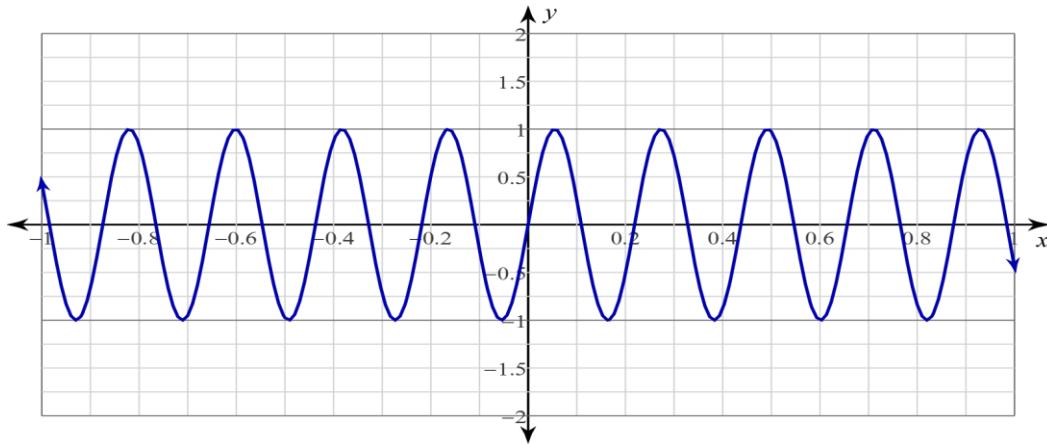
$$y = \text{acos}(bt) \quad \underline{\text{Frequency}} = \frac{|b|}{2\pi} \quad b = 2\pi * \text{frequency} \quad y = \cos(2\pi * f * t)$$

In music, A “tone” is a fixed frequency that is given a name (C, D, E, F, G, A B)

$$C: \quad f_1 = 262 \text{ Hz} \quad D: \quad f_2 = 294 \text{ Hz}$$

$$C: \quad y = \cos(2\pi f_1 t) \quad D: \quad y = \cos(2\pi f_2 t)$$

$$C: \quad y = \cos(524\pi t) \quad D: \quad y = \cos(784\pi t)$$

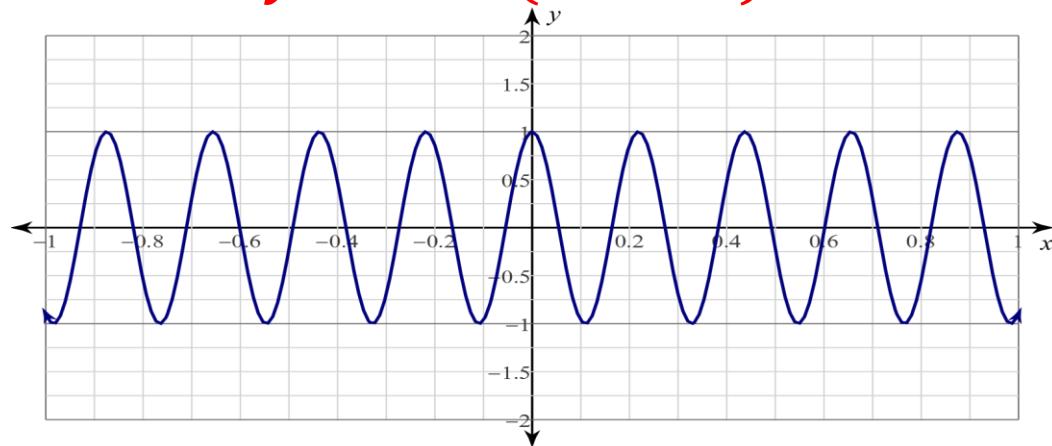


Two different “tones” at the same time combine to form a new frequency called a “beat.”

$$C \text{ and } D \text{ at the same time: } y = \cos(524\pi t) + \cos(784\pi t)$$

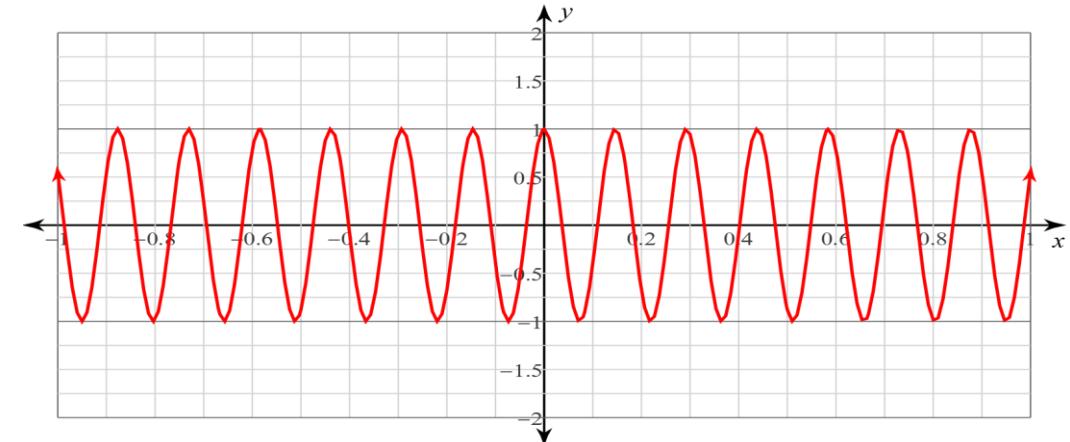
$$C: f_1 = 262 \text{ Hz}$$

$$C: y = \cos(524\pi t)$$

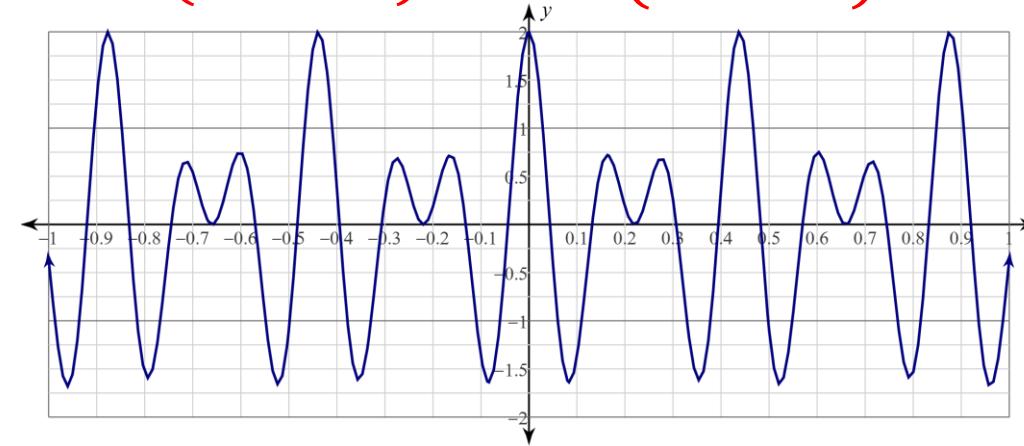
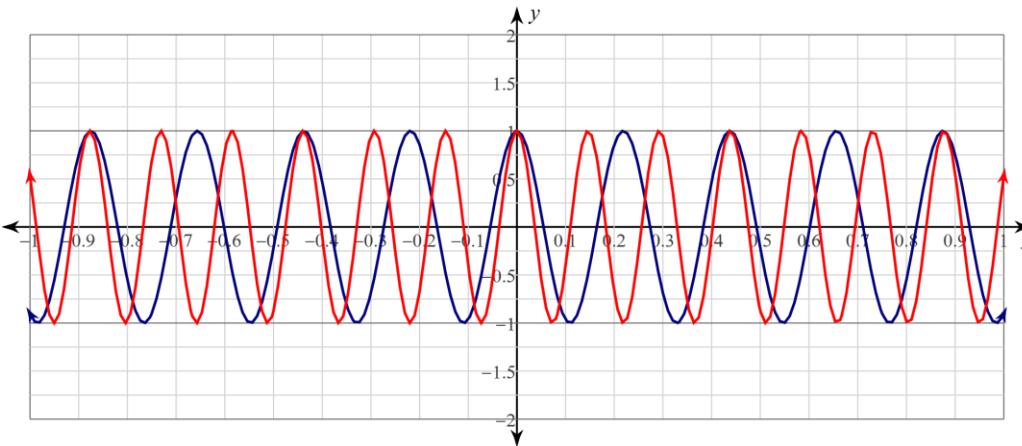


$$D: f_2 = 294 \text{ Hz}$$

$$D: y = \cos(784\pi t)$$



$$C \text{ and } D \text{ at the same time: } y = \cos(524\pi t) + \cos(784\pi t)$$



Two different “tones” at the same time combine to form a new frequency called a “beat.”

C and D at the same time: $y = \cos(524\pi t) + \cos(784\pi t)$

Convert the sum of cosines into a product of cosines

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$y = \cos(524\pi t) + \cos(784\pi t) = 2\cos\left(\frac{524\pi t + 784\pi t}{2}\right)\cos\left(\frac{524\pi t - 784\pi t}{2}\right)$$

$$y = 2\cos(654\pi t)\cos(-130\pi t) \quad y = 2\cos(654\pi t)\cos(130\pi t)$$

Cosine is an even function $\rightarrow \cos(-x) = \cos(x)$

The difference between the 2 frequencies is called the beat frequency: **130 Hz.**