

Math-1060

Lesson 6-2

“Sum and Difference Identities”

The Distributive Property (of multiplication over addition)

$$2(x - 4) \rightarrow 2x - 8$$

Is function notation “distributive”? In other words: $f(a + b) = f(a) + f(b)$

$$f(x) = x^2 \quad f(2) = 4 \quad f(3) = 9 \quad f(2) + f(3) = 13$$

$$f(2) + f(3) \neq f(2 + 3) \quad f(2 + 3) = 25$$

Function notation is NOT “distributive.”

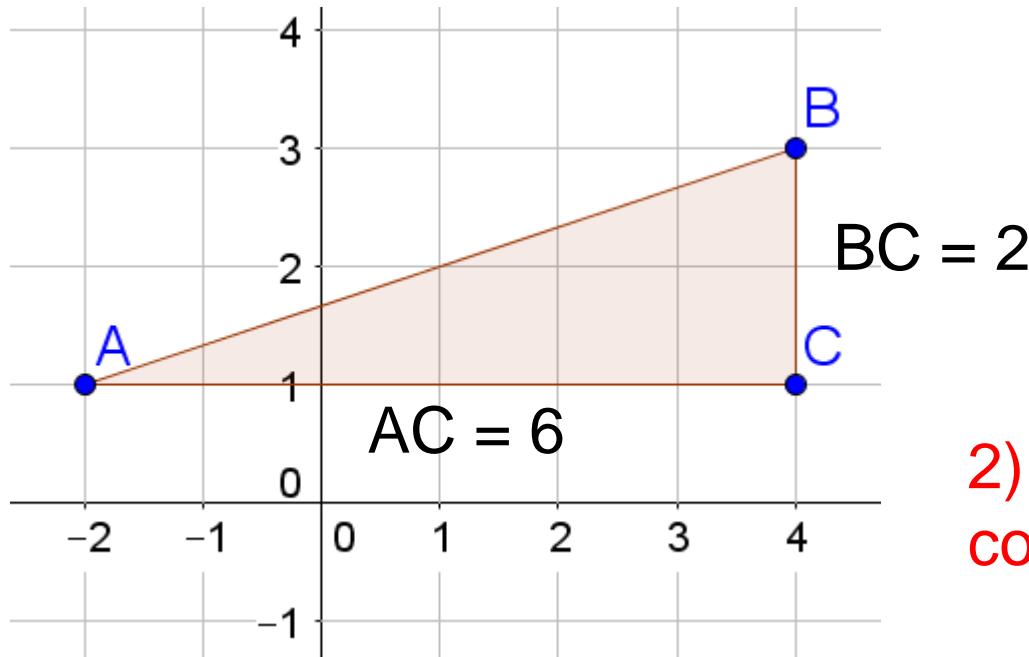
$$\sin(\pi + \frac{\pi}{2}) = \sin\left(\frac{3\pi}{2}\right) = -1 \quad \sin(\pi + \frac{\pi}{2}) \neq \sin \pi + \sin\left(\frac{\pi}{2}\right)$$

$$\sin(\pi) = 0 \quad \sin\left(\frac{\pi}{2}\right) = 1 \quad \boxed{\sin(A + B) \neq \sin A + \sin B}$$

We are now trying to find an identity: $\sin(A + B) = ?$

How can we find the distance between two points (-2, 1) and (4, 3) on the x-y plane?

1) Graphing them on the x-y plane then using the Pythagorean relation.



$$d = \sqrt{(-2 - 4)^2 + (1 - 3)^2}$$

$$d = \sqrt{(-6)^2 + (-2)^2}$$

$$AB = \sqrt{(6)^2 + (2)^2}$$

$$AB = \sqrt{40}$$

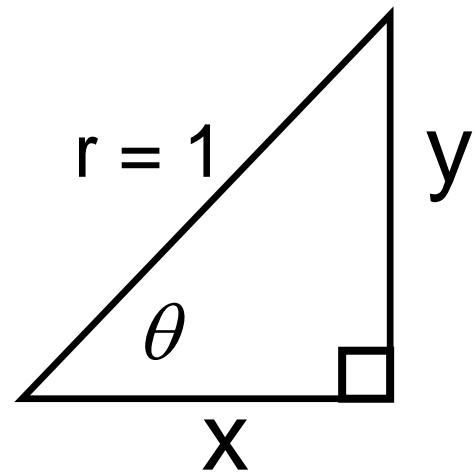
$$AB = 2\sqrt{10}$$

2) Or by using the distance formula (which comes from the Pythagorean relation).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{40} \quad d = 2\sqrt{10}$$

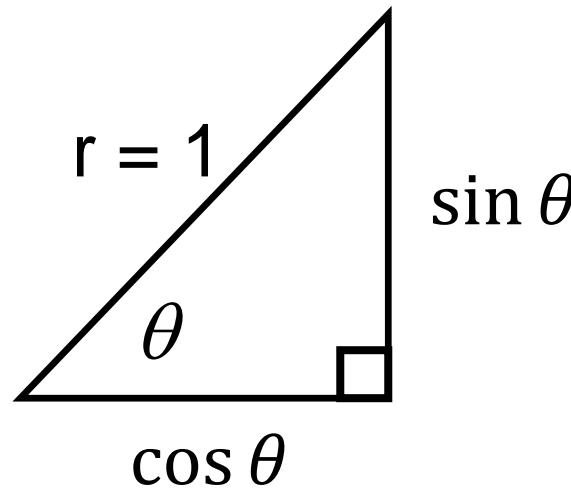
Recall:



$$\sin \theta = \frac{opp}{hyp} = \frac{y}{1} = y$$

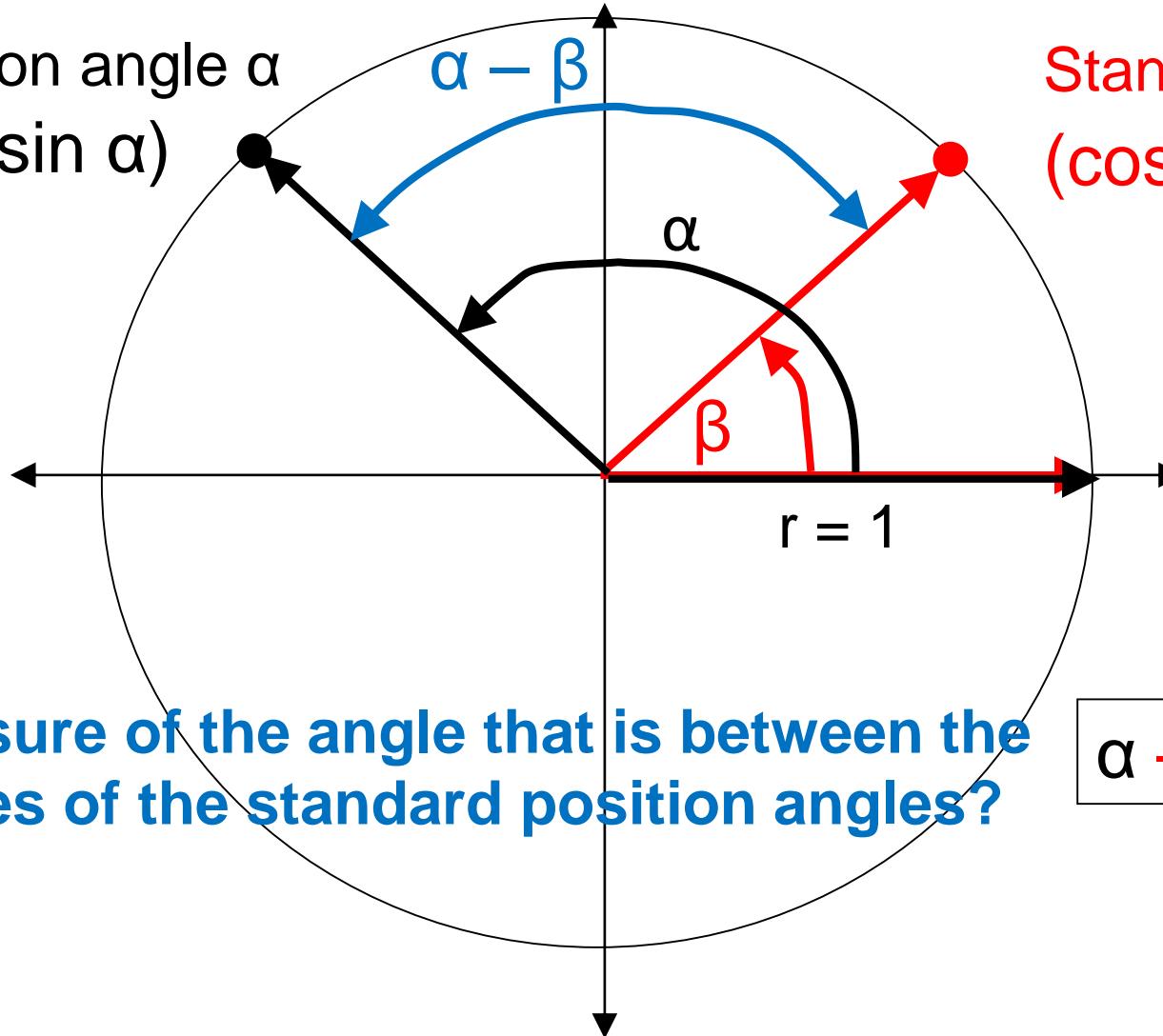
$$\cos \theta = \frac{adj}{hyp} = \frac{x}{1} = x$$

Using Substitution:



Standard position angle α
($\cos \alpha, \sin \alpha$)

Standard position angle β
($\cos \beta, \sin \beta$)

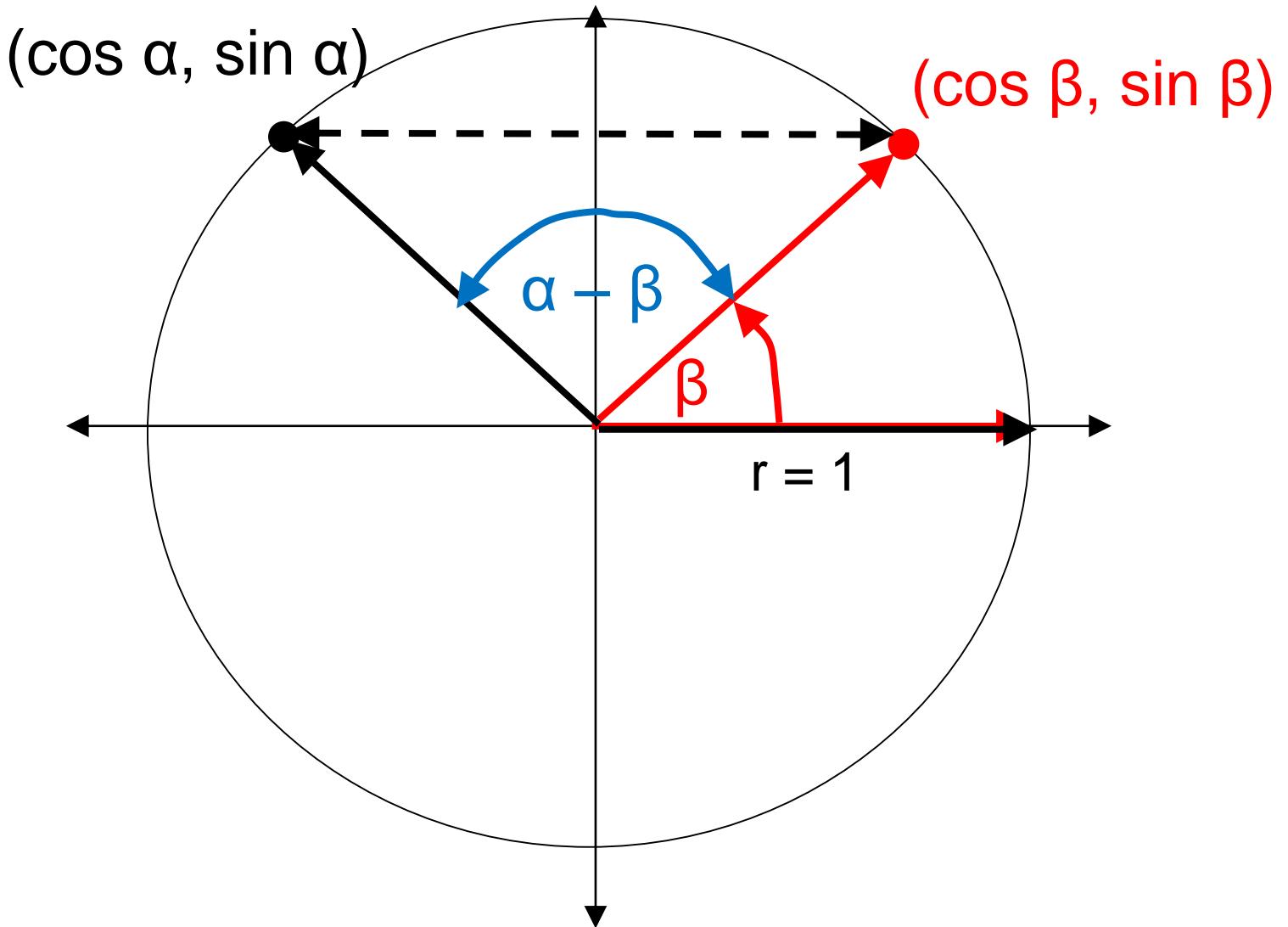


What is the measure of the angle that is between the two terminal sides of the standard position angles?

$\alpha - \beta$

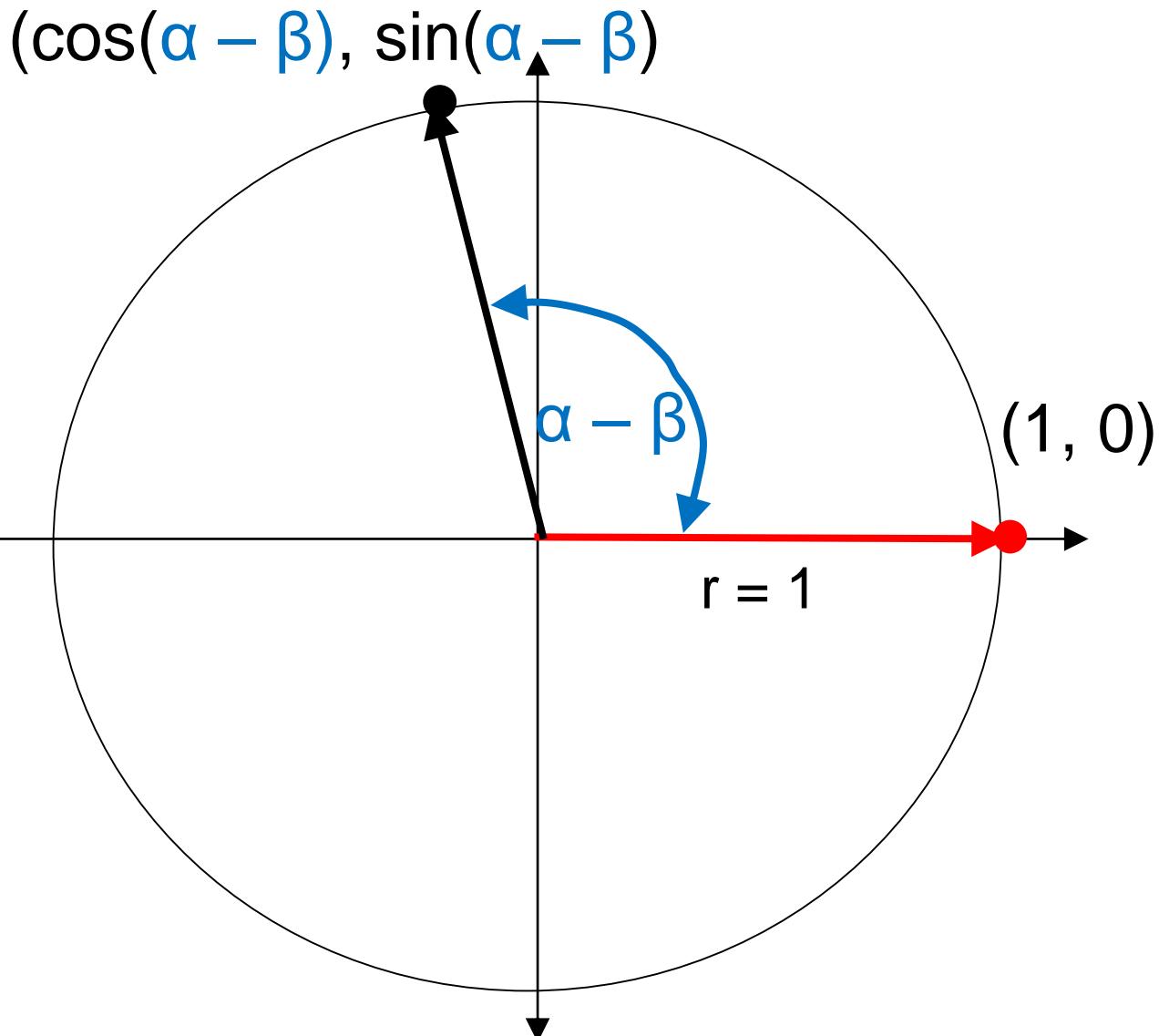
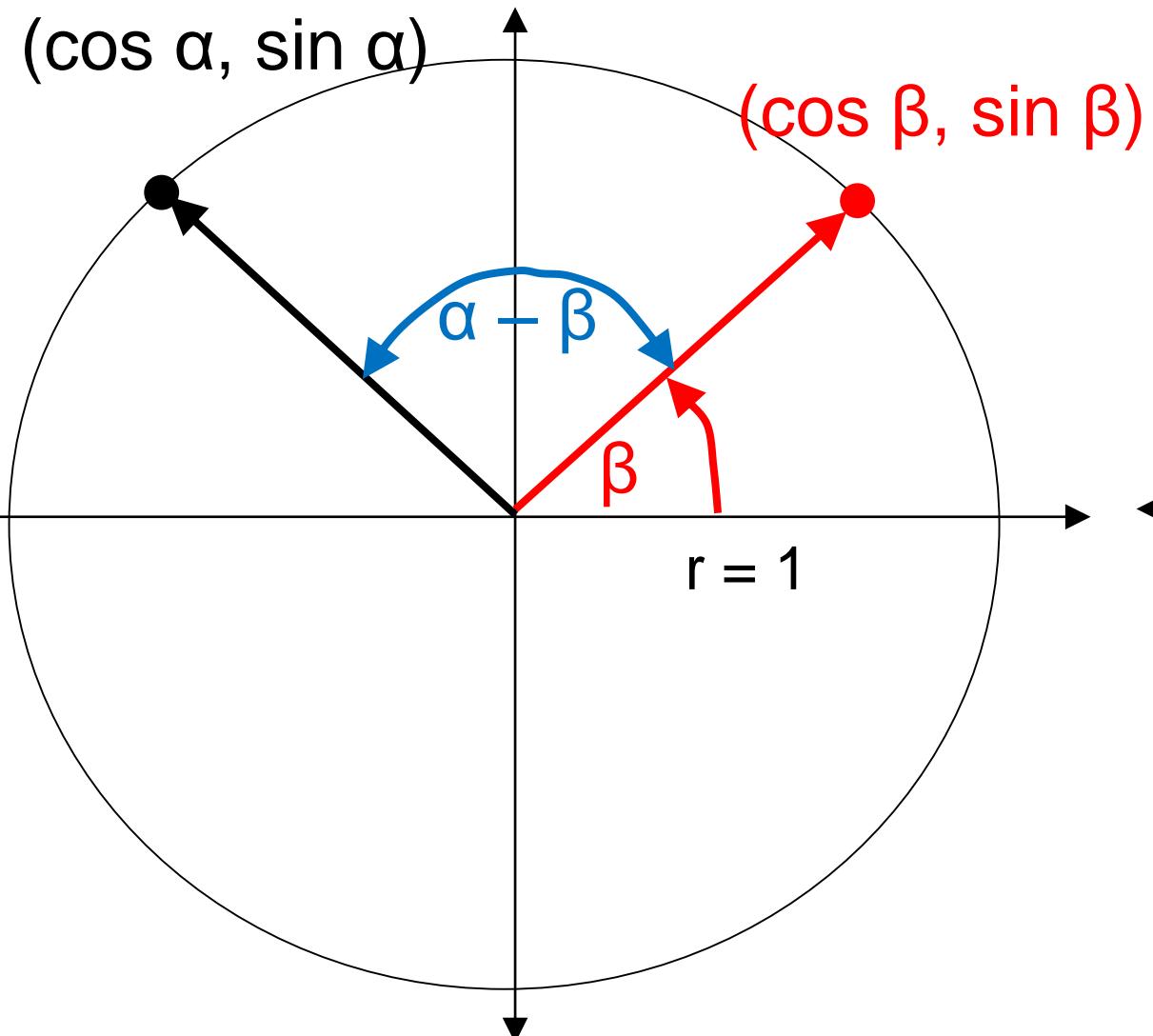
$$d = \sqrt{(\cos \beta - \cos \alpha)^2 + (\sin \beta - \sin \alpha)^2}$$

Find the distance
between these two points.



Rotate the angle (formed by the 2 terminal sides) clockwise by β° .

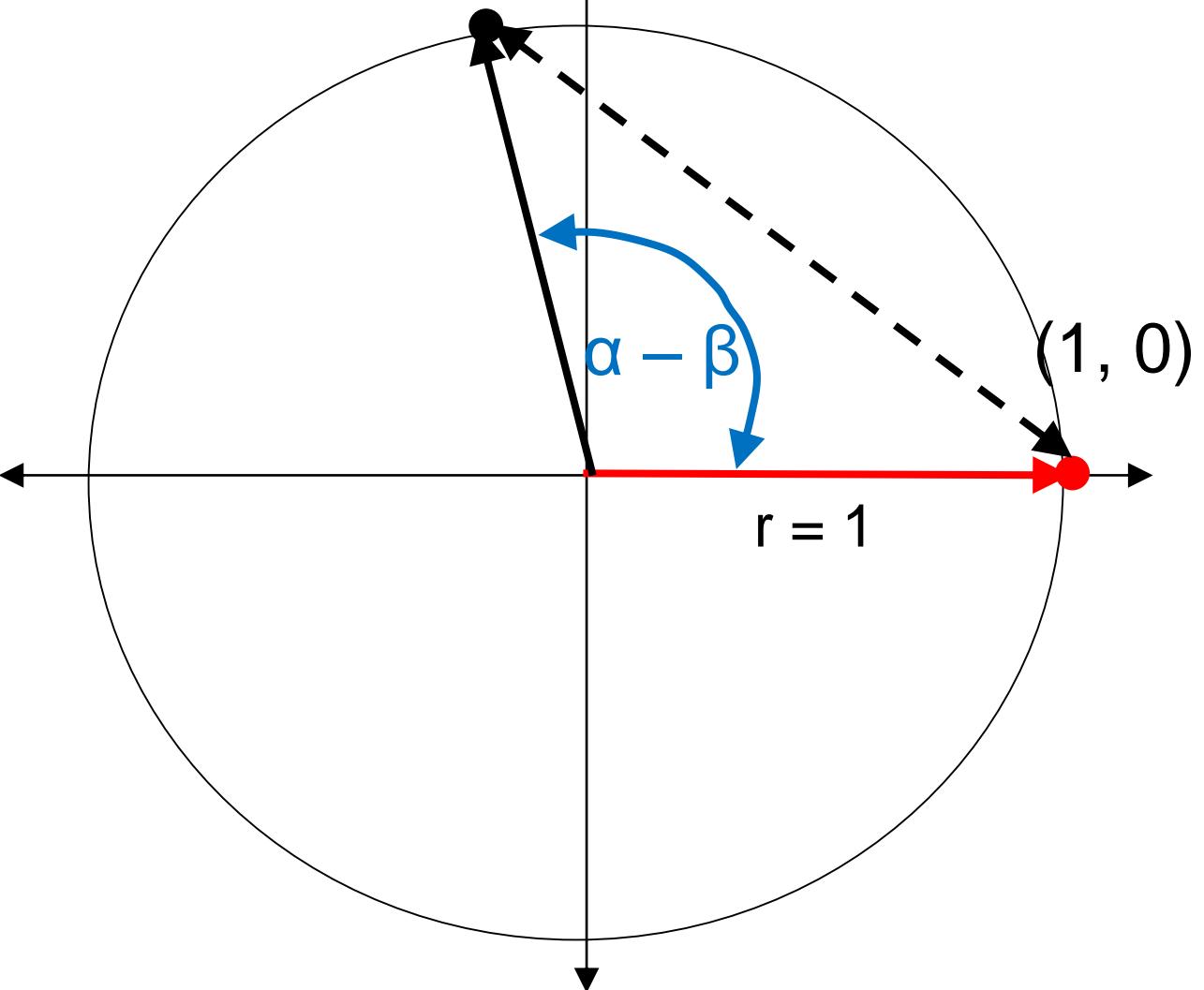
What are the coordinates of the new points?



$$d = \sqrt{(1 - \cos(\beta - \alpha))^2 + (0 - \sin(\beta - \alpha))^2}$$

Find the distance between
these two points.

$$(\cos(\alpha - \beta), \sin(\alpha - \beta))$$



$$d = \sqrt{(\cos \beta - \cos \alpha)^2 + (\sin \beta - \sin \alpha)^2} \quad \text{Simplify}$$

$$d = \sqrt{\cos^2 \beta - 2 \cos \beta \cos \alpha + \cos^2 \alpha + \sin^2 \beta - 2 \sin \beta \sin \alpha + \sin^2 \alpha}$$

$$d = \sqrt{\cos^2 \beta + \sin^2 \beta - 2 \cos \beta \cos \alpha + \cos^2 \alpha + \sin^2 \alpha - 2 \sin \beta \sin \alpha}$$

$$d = \sqrt{2 - 2 \cos \beta \cos \alpha - 2 \sin \beta \sin \alpha}$$

$$d = \sqrt{(1 - \cos(\beta - \alpha))^2 + (0 - \sin(\beta - \alpha))^2} \quad \text{Simplify}$$

$$d = \sqrt{1 - 2 \cos(\beta - \alpha) + \cos^2(\beta - \alpha) + \sin^2(\beta - \alpha)}$$

$$d = \sqrt{2 - 2 \cos(\beta - \alpha)} \quad \text{The distances are equal to each other.}$$

$$\sqrt{2 - 2 \cos \beta \cos \alpha - 2 \sin \beta \sin \alpha} = \sqrt{2 - 2 \cos(\beta - \alpha)}$$

The distances are equal to each other.

$$\sqrt{2 - 2 \cos \beta \cos \alpha - 2 \sin \beta \sin \alpha} = \sqrt{2 - 2 \cos(\beta - \alpha)}$$

Square both sides and simplify

$$2 - 2 \cos \beta \cos \alpha - 2 \sin \beta \sin \alpha = 2 - 2 \cos(\beta - \alpha)$$

$$\cos \beta \cos \alpha + \sin \beta \sin \alpha = \cos(\beta - \alpha)$$

Re-arrange so it's easier to remember

$$\cos(\beta - \alpha) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Cosine is even so: $\cos(-x) = \cos(x)$

$$\boxed{\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta}$$

Presto: we have the “Difference Identity”

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Now the Sum Identity

$$\cos(\alpha - (-\beta)) = \cos \alpha \cos(-\beta) + \sin \alpha \sin(-\beta)$$

Cosine is even so: $\cos(-x) = \cos(x)$

Sine is odd so: $\sin(-x) = -\sin(x)$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

What are the “nice” reference angles?

$$30 = \frac{\pi}{6} \quad 45 = \frac{\pi}{4} \quad 60 = \frac{\pi}{3} \quad 90 = \frac{\pi}{2}$$

Just look at the denominator, what is the reference angle?

$$30^\circ = \frac{5\pi}{6}$$

$$45^\circ = \frac{7\pi}{4}$$

$$60^\circ = \frac{4\pi}{3}$$

$$90^\circ = \frac{3\pi}{2}$$

Cosine Difference Identity

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Cosine Sum Identity

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

Using these identities, we can find the exact trig ratios for more angles.

Rewrite the angle as the sum of “nice” angles. $\cos \frac{5\pi}{12}$

$$\rightarrow \cos\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right) \rightarrow \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \rightarrow \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{6}\right)$$

$$\rightarrow \frac{\sqrt{2}}{2} * \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} * \frac{1}{2} \rightarrow \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \rightarrow \boxed{\frac{\sqrt{6} - \sqrt{2}}{4}}$$

Simplify. $\cos(2x) \cos(7x) - \sin(2x) \sin(7x)$

$$\rightarrow \cos(2x + 7x)$$

Find the exact trig ratios for these angles.

$$\cos \frac{7\pi}{12} \quad \cos(15)$$

$$\rightarrow \cos\left(\frac{10\pi}{12} - \frac{3\pi}{12}\right) \rightarrow \cos\left(\frac{5\pi}{6} + \frac{\pi}{4}\right) \rightarrow \cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{5\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$$

$$\rightarrow \frac{-\sqrt{3}}{2} * \frac{\sqrt{2}}{2} + \frac{1}{2} * \frac{\sqrt{2}}{2} \rightarrow \frac{-\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \boxed{\rightarrow \frac{\sqrt{2} - \sqrt{6}}{4}}$$

$$\rightarrow \cos(45 - 30) \rightarrow \cos(45)\cos(30) + \sin(45)\sin(30)$$

$$\rightarrow \frac{\sqrt{2}}{2} * \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} * \frac{1}{2} \rightarrow \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \boxed{\rightarrow \frac{\sqrt{6} + \sqrt{2}}{4}}$$

Derive the Sine Sum/Difference Identities

$$\sin \theta = \cos(90 - \theta)$$

Cofunction Identity

$$\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

Let $\Theta = (A + B)$

$$\sin(A + B) = \cos(90 - (A + B))$$

Expand using the Cosine Difference Identity

$$\sin(A + B) = \cos(90 - A)\cos B + \sin(90 - A)\sin B$$

Substitution

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Sine Sum Identity

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Sine Difference Identity

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \cdot \tan v}$$

Find the exact trig ratios for these angles.

$$\sin \frac{5\pi}{12} \quad \sin(15)$$

$$\rightarrow \sin\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right) \rightarrow \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \rightarrow \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

$$\rightarrow \frac{\sqrt{2}}{2} * \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} * \frac{1}{2} \rightarrow \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \frac{\sqrt{2}}{2} \boxed{\rightarrow \frac{\sqrt{6} + \sqrt{2}}{4}}$$

$$\rightarrow \sin(60 - 45) \rightarrow \sin(60)\cos(45) - \cos(60)\sin(45)$$

$$\rightarrow \frac{\sqrt{3}}{2} * \frac{\sqrt{2}}{2} - \frac{1}{2} * \frac{\sqrt{2}}{2} * \rightarrow \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \boxed{\rightarrow \frac{\sqrt{6} - \sqrt{2}}{4}}$$

Using combinations of the following angles: 30, 45, 60, 90;
find three other angles for which an exact trig ratio is possible
using the Sum-Difference Identities.

$$45 - 30 = 15 \quad 45 + 30 = 75 \quad 45 + 60 = 105$$

Using combinations of the following angles: $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ find three
other angles for which an exact trig ratio is possible using the
Sum-Difference Identities.

$$\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$$

Tangent Sum Identity

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Multiply numerator and denominator by:

$$\frac{1}{\cos A \cos B}$$

$$\begin{aligned} \tan(A + B) &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\ &= \frac{\cancel{\sin A \cos B}}{\cancel{\cos A \cos B}} + \frac{\cancel{\cos A \sin B}}{\cancel{\cos A \cos B}} \\ &\quad - \frac{\cancel{\cos A \cos B}}{\cancel{\cos A \cos B}} - \frac{\cancel{\sin A \sin B}}{\cancel{\cos A \cos B}} \end{aligned}$$

$$\tan(A + B) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A \sin B}{\cos A \cos B}}$$

Tangent Sum Identity

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A * \tan B}$$

$$\tan(A - B) = \tan(A + (-B))$$

Tangent Sum Identity

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A * \tan B}$$

$$\tan(A + (-B)) = \frac{\tan A + \tan(-B)}{1 - \tan A * \tan(-B)}$$

Tangent is odd so: $\tan(-x) = -\tan(x)$

Tangent Difference Identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A * \tan B}$$

On the reference sheet the Sum/Difference Identities are given as:

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

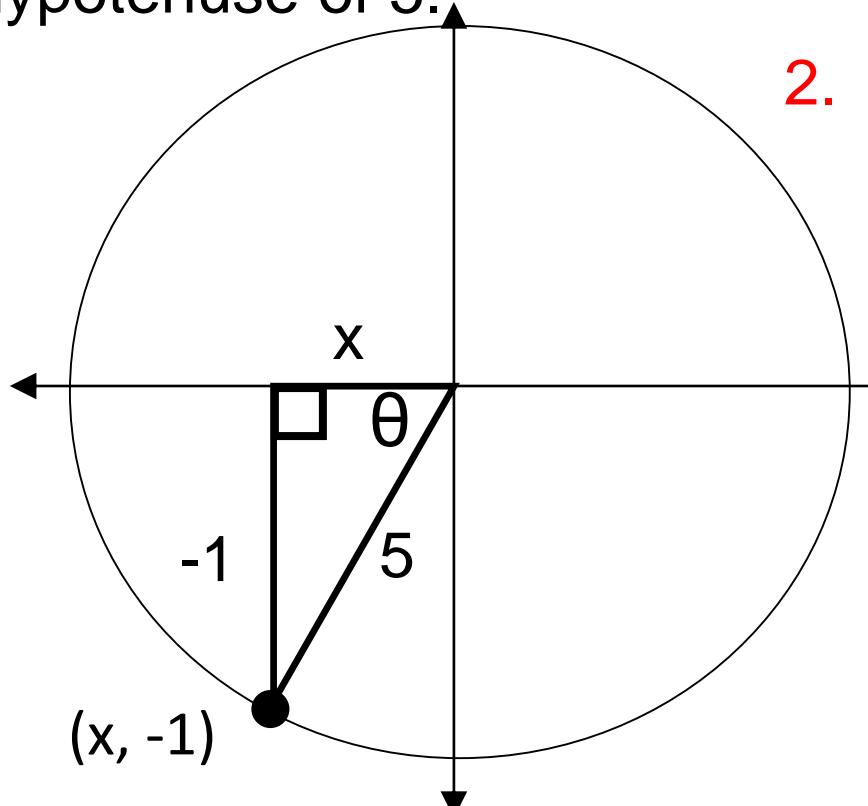
$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \cdot \tan v}$$

Find the exact value of $\tan(A + B)$ given the following information:

$$\sin A = -\frac{1}{5} \quad \cos B = \frac{1}{3}$$

The terminal side of 'A' is in quadrant III and the terminal side of 'B' is quadrant IV.

1. Graph the standard position angle 'A' then build a right triangle with the reference angle " θ " (the angle whose terminal side passes through $(x, -1)$) that has a hypotenuse of 5.



2. Find 'x': $x^2 + (-1)^2 = 25$

$$x = -\sqrt{24} \quad x = -2\sqrt{6}$$

3. Find $\tan A$

$$\tan A = \frac{y}{x}$$

$$\tan A = \frac{\sqrt{6}}{12}$$

$$\tan A = \frac{-1}{-2\sqrt{6}}$$

$$\cos B = \frac{1}{3}$$

4. Graph the standard position angle 'B' then build a right triangle with the reference angle being " Θ " (the angle whose terminal side passes through $(1, y)$) that has a hypotenuse of 3.

5. Find 'y':

$$y^2 + (1)^2 = 9$$

$$y = -\sqrt{8}$$

$$y = -2\sqrt{2}$$

6. Find tanB

$$\tan B = \frac{y}{x}$$

$$\tan A = \frac{-2\sqrt{2}}{1}$$

$$\tan A = -2\sqrt{2}$$

7. Find tan(A + B)

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A + B) = \frac{\frac{\sqrt{6}}{12} + (-2\sqrt{2})}{1 - \frac{\sqrt{6}}{12} * +(-2\sqrt{2})}$$

$$= \frac{\sqrt{6} - 24\sqrt{2}}{12 - 4\sqrt{3}}$$

