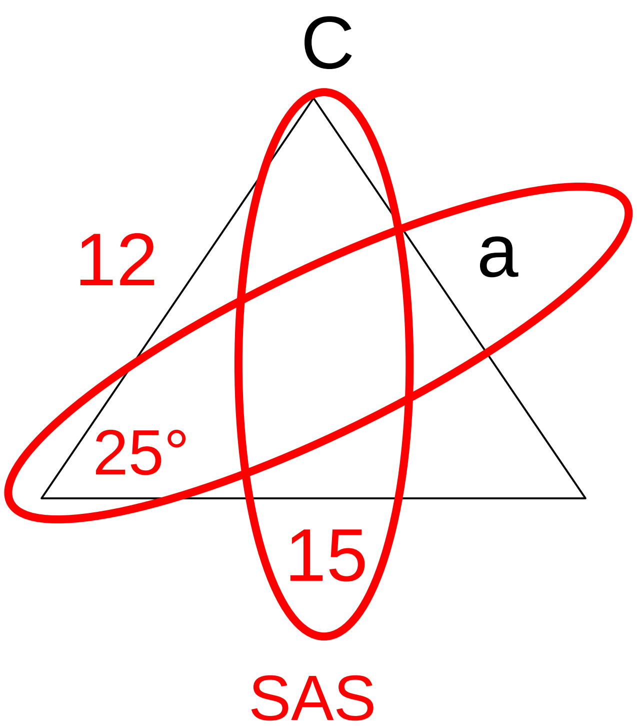
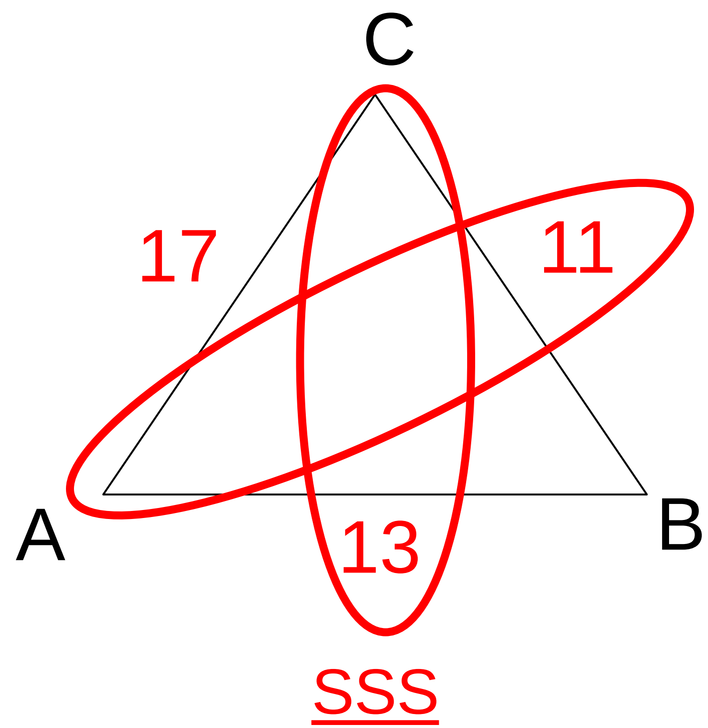
Math-1060

Lesson 4-5

The Law of Cosines

Solve using Law of Sines.





Every pair of loops will have 2 unknowns.

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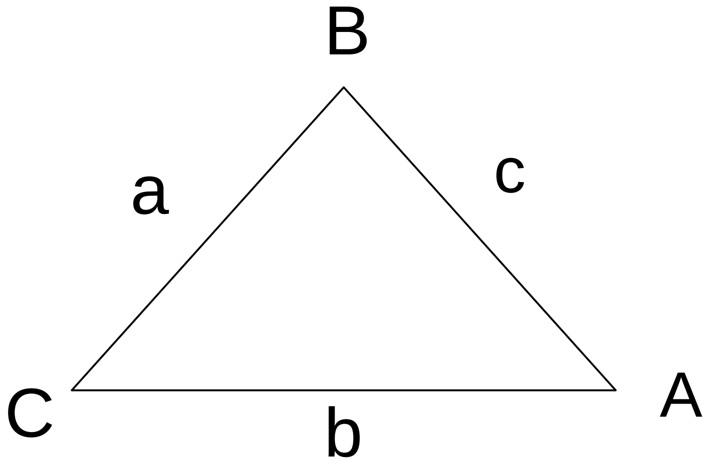
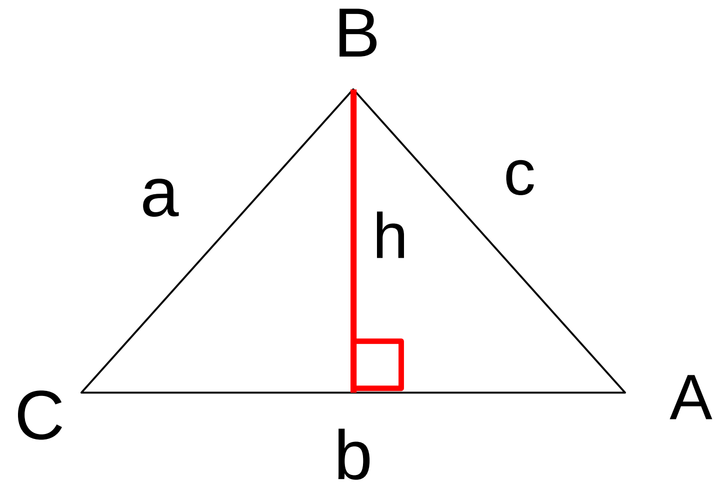
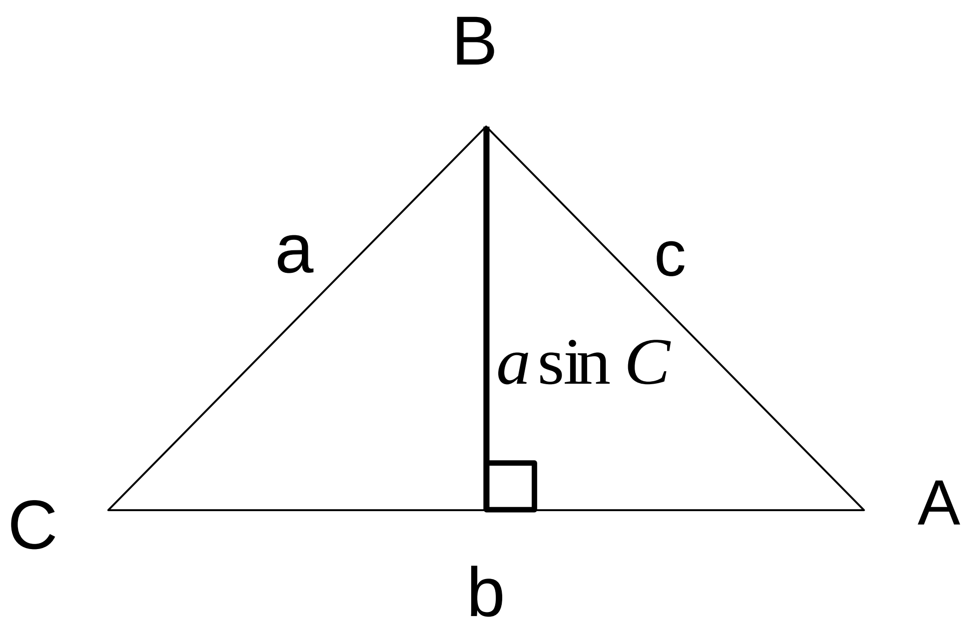
We need another equation.





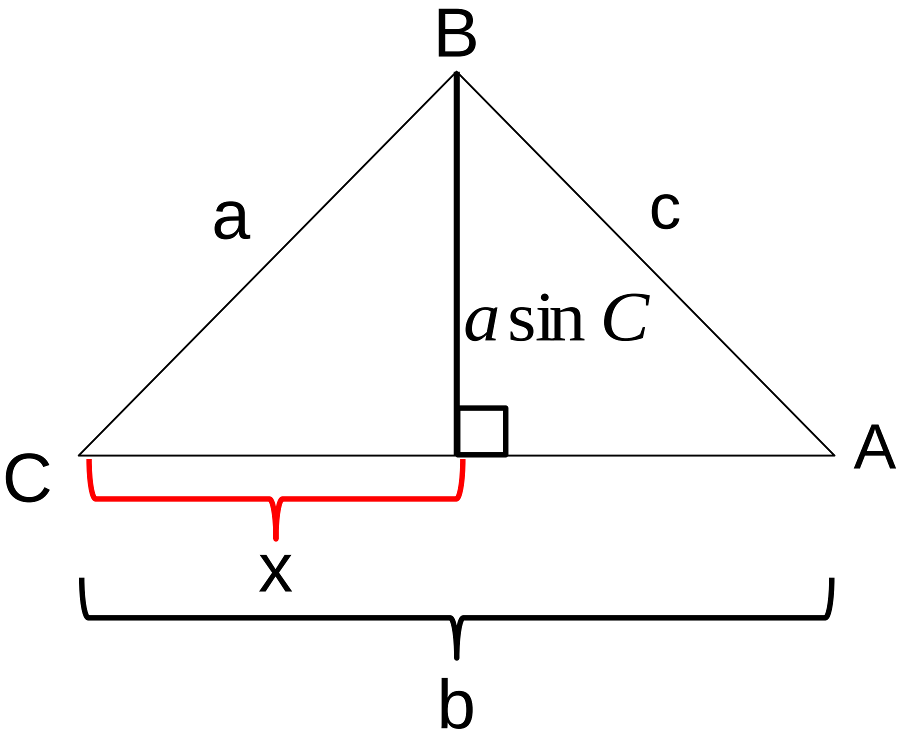
“Drop” and altitude from Vertex B to the opposite side.

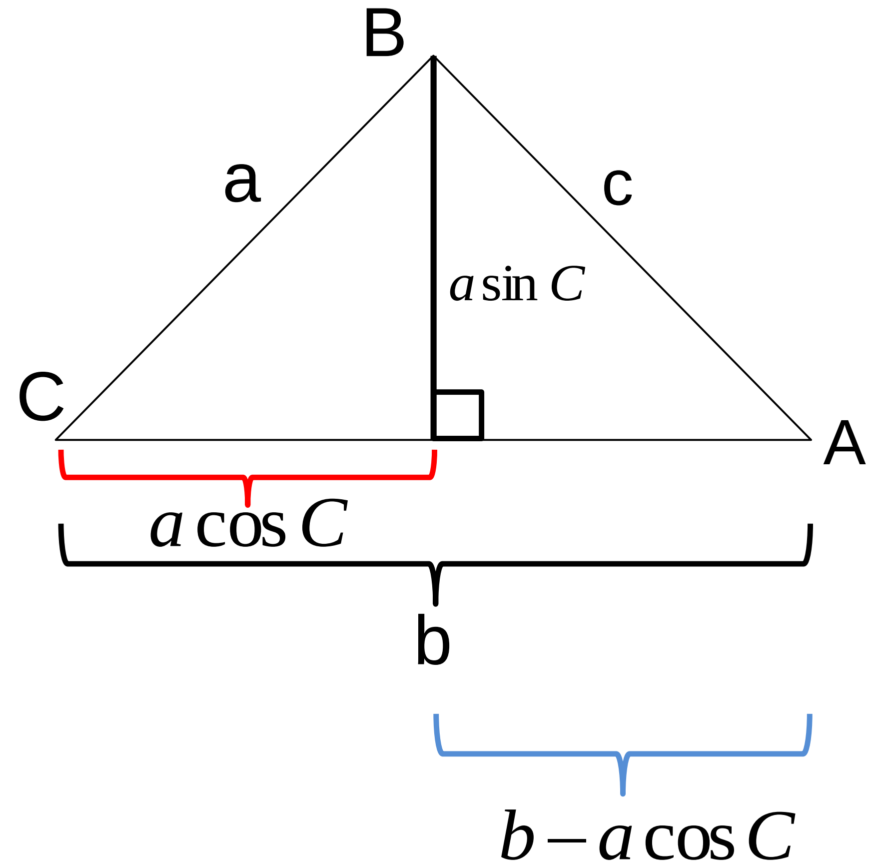


$$\sin C = \frac{h}{a}$$

$$h = a \sin C$$

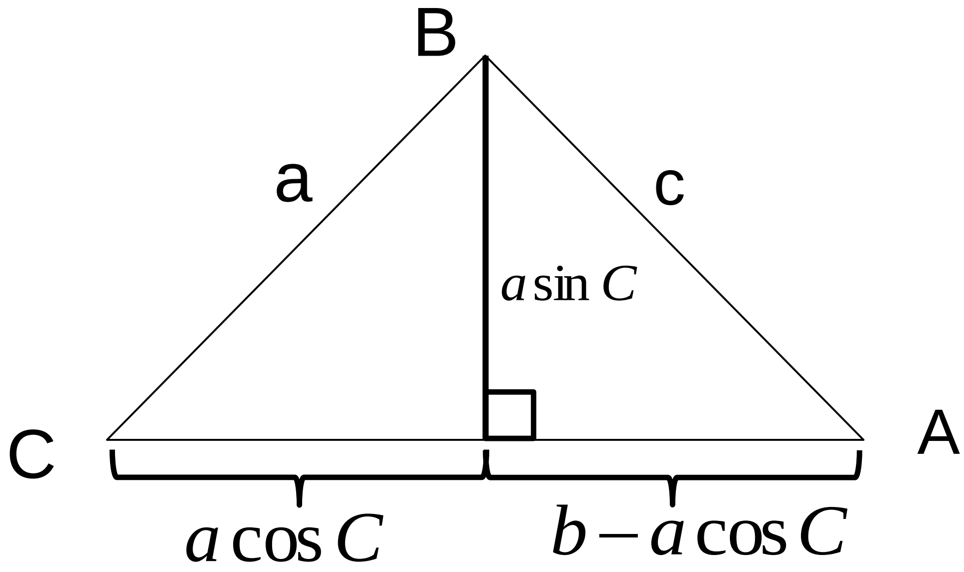
Substitute $a \sin C$ for ‘h’ in the figure.





$$\cos C = \frac{x}{a}$$

$$x = a\cos C$$



Using Pythagorean Theorem on the right-side triangle:

$$c^2 = (a\sin C)^2 + (b - a\cos C)^2$$

$$c^2 = a^2\sin^2 C + b^2 - 2ab\cos C + a^2\cos^2 C$$

$$c^2 = a^2(\sin^2 C + \cos^2 C) + b^2 - 2ab\cos C$$

$$\sin^2 C + \cos^2 C = 1$$

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$$

Law of Cosines: Uses 3 sides and 1 angle.

Law of Cosines

Let ΔABC be any triangle with sides and angles labeled in the usual way. Then

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad A = \cos^{-1}\left(\frac{a^2 - b^2 - c^2}{-2bc}\right)$$

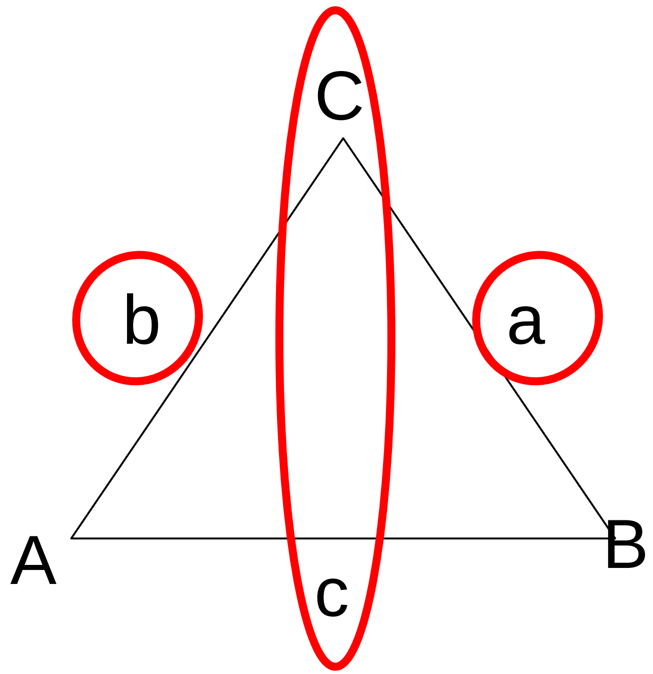
$$b^2 = a^2 + c^2 - 2ac\cos B \qquad B = \cos^{-1}\left(\frac{b^2 - a^2 - c^2}{-2ac}\right)$$

Pythagorean Theorem

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad C = \cos^{-1}\left(\frac{c^2 - a^2 - b^2}{-2ab}\right)$$

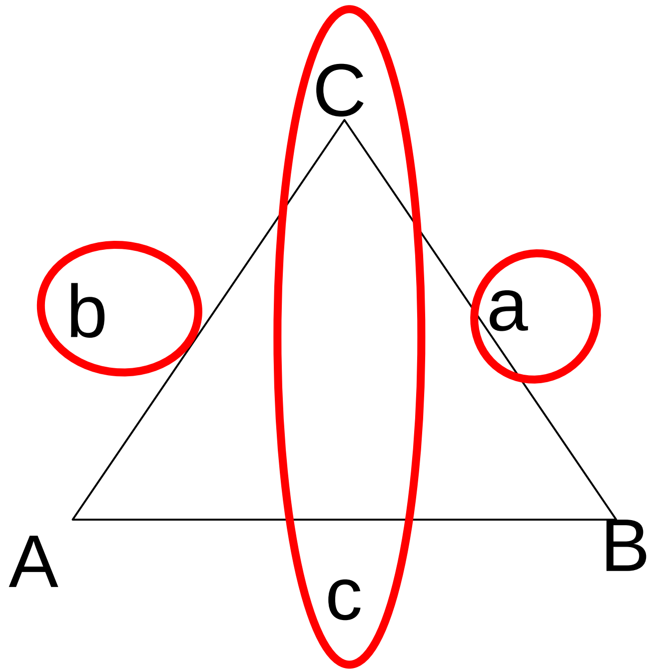
There is a pattern for Law of Cosines

1. Label the triangle.
2. Loop the 1 side and its opposite angle and the 2 other sides.



If 3 of the 4 values are known, use Law of Cosines.

Law of Cosines



$$c^2 = a^2 + b^2 - 2ab\cos C$$

Notice in the pattern that
Angle "C"
is opposite ends of the equation from side length 'c'.

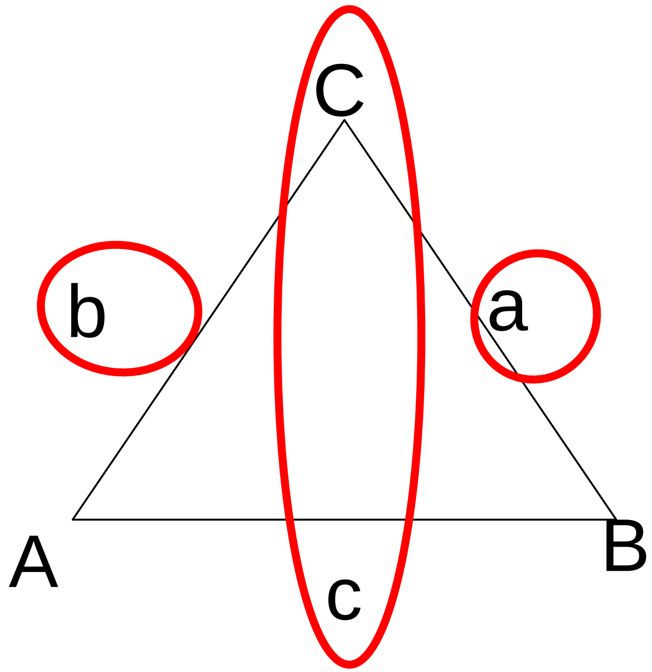
$\cos 90 = ?$ = 0

If C = 90° (right triangle) $c^2 = a^2 + b^2 - 2ab\cos C$ becomes

$$c^2 = a^2 + b^2$$

Is that familiar?

Law of Cosines: Patterns in the formula



$$c^2 = a^2 + b^2 - 2ab\cos C$$

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There are some "<u>Gotcha's</u>" with the Law of Cosines (when there is an ***<u>obtuse angle</u>***).

Law of Cosines will always be the 1st step for SSS and SAS.

When solving for the other sides and angles, you can choose between <u>Law of Cosines</u> (more difficult calculation) or <u>Law of Sines</u> (easier calculation).

sin(135) = ? $= \sqrt{2}/2 \approx 0.7071$ sin(45) = ? $= \sqrt{2}/2 \approx 0.7071$

→ When using the Law of Sines, the <u>calculator will give the smallest angle</u> even if the angle is actually obtuse.

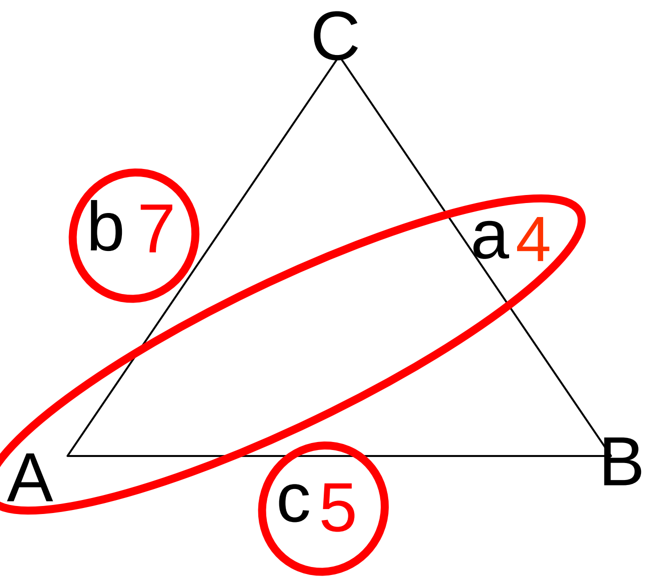
→If you use <u>Law of Cosines</u> for follow-on steps, there is no ambiguity. You'll get the correct angle.

Cos(135) = ? $= -\sqrt{2}/2 \approx -0.7071$ Cos(45) = ? $= \sqrt{2}/2 \approx 0.7071$

→ If you choose to use the Law of Sines, <u>find the smallest angle 1st.</u>

For SSS, you can find any angle 1st.

$$a^2 = b^2 + c^2 - 2bc\cos A$$



$$a^2 = b^2 + c^2 - 2bc\cos A$$

Replace letters with parentheses.

$$(\ \ )^2 = (\ \ )^2 + (\ \ )^2 - 2(\ \ )(\ \ )\cos(\ \ )$$

Substitute numbers into the parentheses.

$$(4)^2 = (7)^2 + (5)^2 - 2(7)(5)\cos(A)$$

$$a = 4 \quad b = 7 \quad c = 5$$

$$16 = 49 + 25 - 70\cos A$$

$$16 = 74 - 70\cos A$$

Can we subtract 70 from 74?

$$\frac{16-74}{-70} = \cos A$$

$$0.8286 = \cos A$$

$$\cos^{-1}(0.8286) = A \quad \boxed{= 34^\circ}$$

You can use Law of Cosines Next

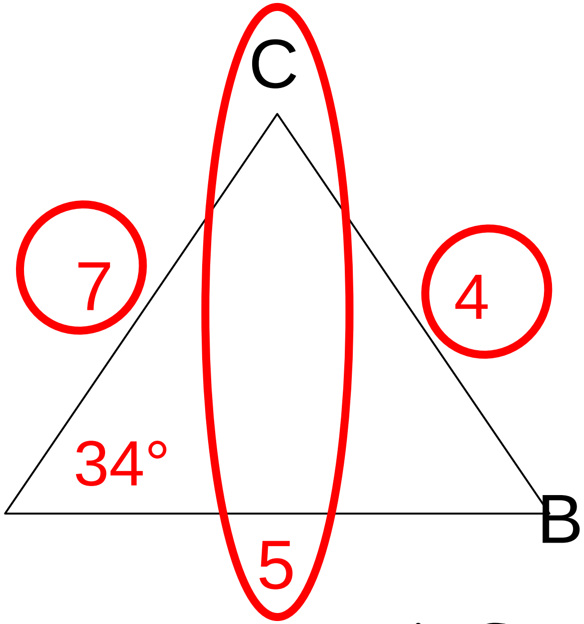
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$(\ )^2 = (\ )^2 + (\ )^2 - 2(\ )(\ )\cos(\ )$$

$$(5)^2 = (7)^2 + (4)^2 - 2(7)(4)\cos(C)$$

$$\frac{(5^2 - 7^2 - 4^2)}{(-2*7*4)} = \cos C \qquad 0.7143 = \cos C$$

$$\cos^{-1}(0.7143) = C \qquad \boxed{= 44.4^\circ}$$



Or Law of Sines

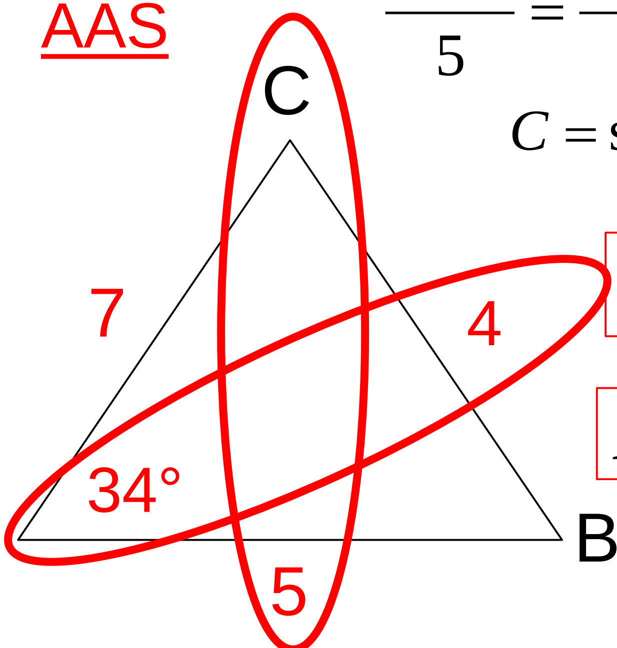
AAS

$$\frac{\sin C}{5} = \frac{\sin 34}{4}$$

$$C = \sin^{-1}\left(\frac{5\sin 34}{4}\right)$$

$$\boxed{C = 44.4^\circ}$$

$$\boxed{B = 101.6}$$



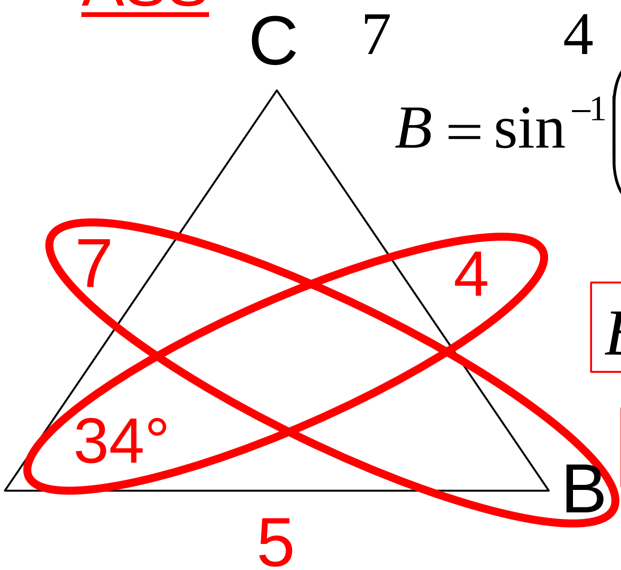
ASS

$$\frac{\sin B}{7} = \frac{\sin 34}{4}$$

$$B = \sin^{-1}\left(\frac{7\sin 34}{4}\right)$$

$$\boxed{B = 78.1^\circ}$$

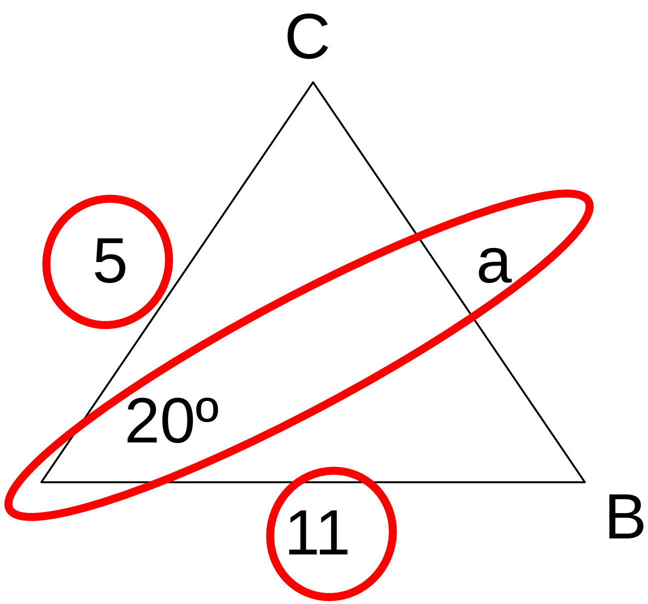
$$\boxed{C = 67.9}$$



Why are they different?

For SAS, you must find the missing side 1st.

$$A = 20^\circ \quad b = 5 \quad c = 11$$



(1) If it is not already given, draw and label a triangle.

(2) What pattern? Law of Sines ? Or Law of Cosines?

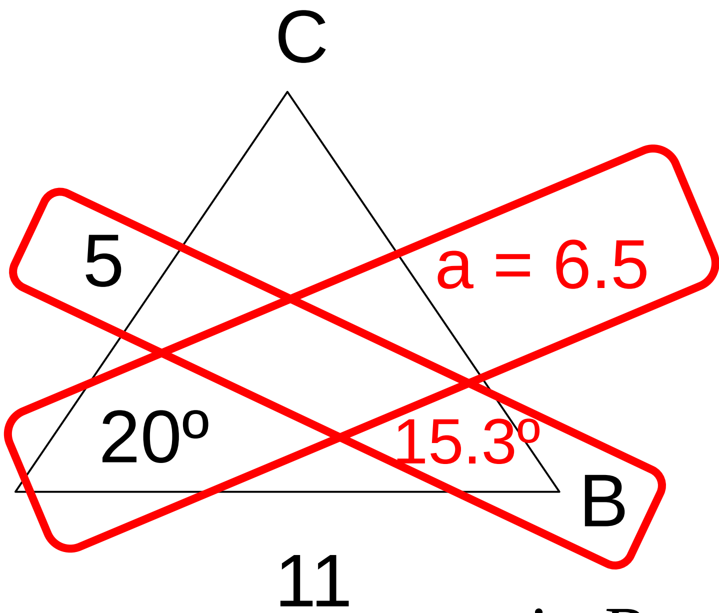
(3) Use Law of Cosines

$$a^2 = 5^2 + 11^2 - 2(5)(11)\cos 20$$

$$a^2 = 25 + 121 - 110\cos 20$$

$$a^2 = 42.6$$

$$a = 6.5$$



(4) Which angle do you find next?

Law of Sines is easier.

Find the **smallest angle first!!!!)**

$$\frac{\sin B}{b} = \frac{\sin A}{a} \qquad \frac{\sin B}{5} = \frac{\sin 20}{6.5}$$

$$\sin B = \frac{5\sin 20}{6.5} \qquad B = \sin^{-1}\left(\frac{5\sin 20}{6.5}\right) \qquad \boxed{B = 15.3^\circ}$$

(5) Triangle Sum Theorem

$$m\angle A + m\angle B + m\angle C = 180$$

$$m\angle C = 180 - 20^\circ - 15.3^\circ$$

$$m\angle C = 144.7^\circ$$

What angle would Law of Sines give you if you found Angle C in step (4) above?

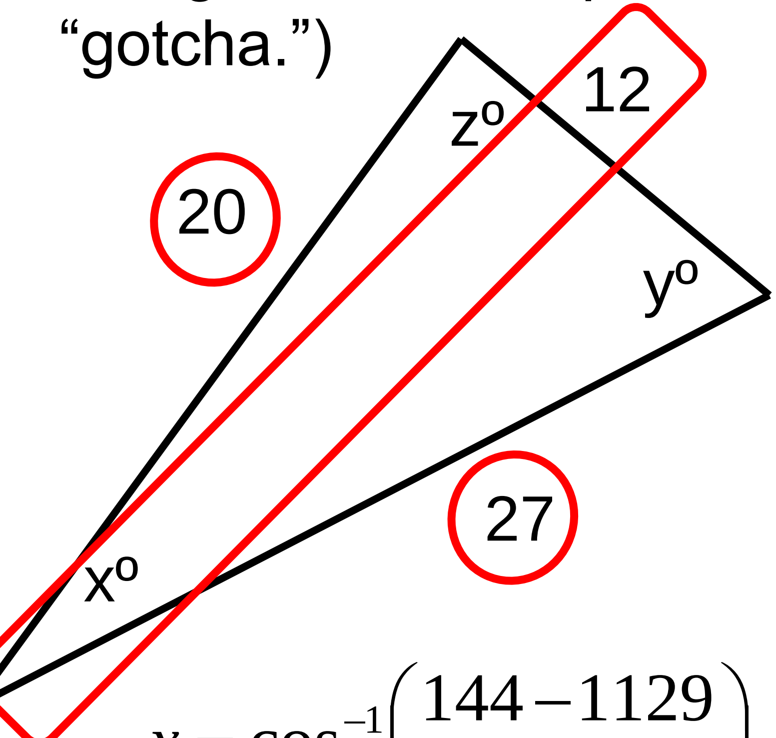
$$180 - \boxed{144.7} = 35.3$$

These two angles have the same reference angle.

Which is the correct angle?

For the SSS case: find the largest angle first!

(This will get the obtuse angle "out in the open" if it exists in the triangle, so subsequent use of Law of Sines won't have the "gotcha.")



If you don't find the obtuse angle 1st you have to remember to find the smallest angle next when you use the Law of Sines.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$(12)^2 = (20)^2 + (27)^2 - 2(20)(27)\cos(x)$$

$$144 = 400 + 729 - 1080\cos(x)$$

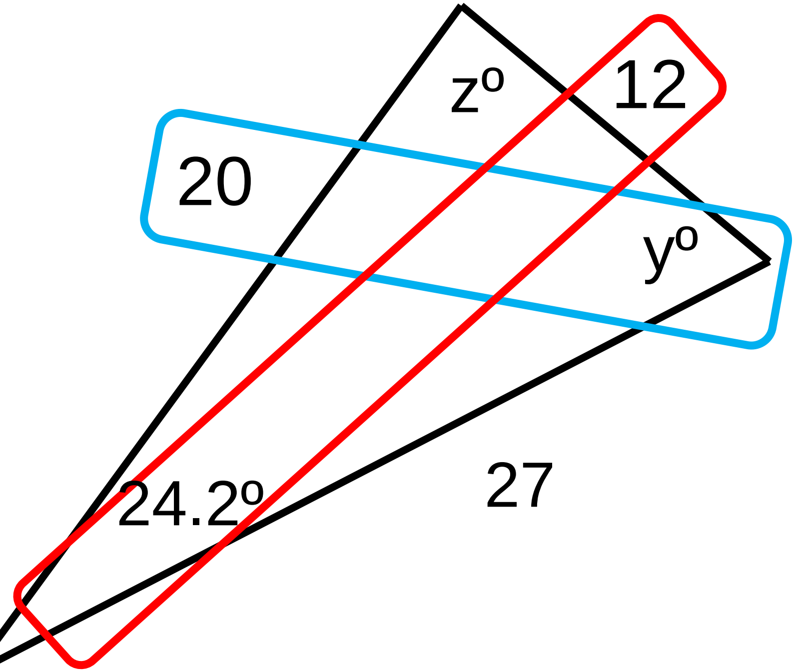
$$144 = 1129 - 1080\cos(x)$$

$$x = \cos^{-1}\left(\frac{144 - 1129}{-1080}\right)$$

$$x = \cos^{-1}\left(\frac{144 - 1129}{-1080}\right)$$

$$x = 24.2^\circ$$

We didn't find the obtuse angle 1st using Law of Cosines so, if you use the Law of Sines to find the 2nd angle, find the ***SMALLEST ANGLE*** next.



$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \frac{\sin y^\circ}{20} = \frac{\sin 24.2}{12}$$

$$\sin y = \frac{20\sin 24.2}{12}$$

$$y = \sin^{-1}\left(\frac{20\sin 24.2}{12}\right) \qquad y = 43.1^\circ$$

$$z = 180 - 24.2 - 43.1^\circ \qquad z = 112.7^\circ$$

What angle would Law of Sines give you if you found Angle z above? $180 - \boxed{112.7} = 67.3$

These two angles have the same reference angle.

Which is the correct angle?

Altitude of a triangle: The perpendicular distance from any vertex to its opposite side.

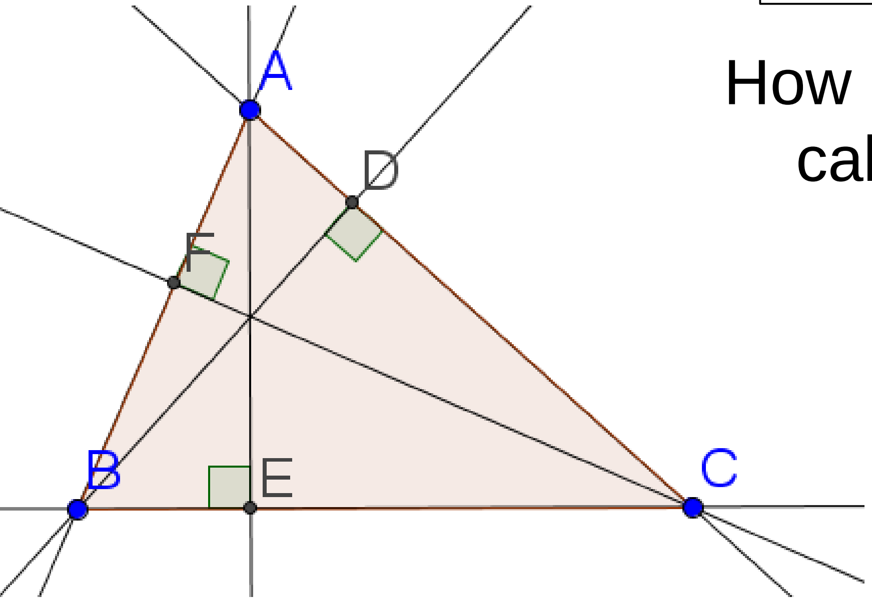
Altitude of a triangle: means the same thing as the height of a triangle. Height = Altitude

How many "heights" (altitudes) does a triangle have? three

base of a triangle: any side of a triangle. base = side

Height = Altitude

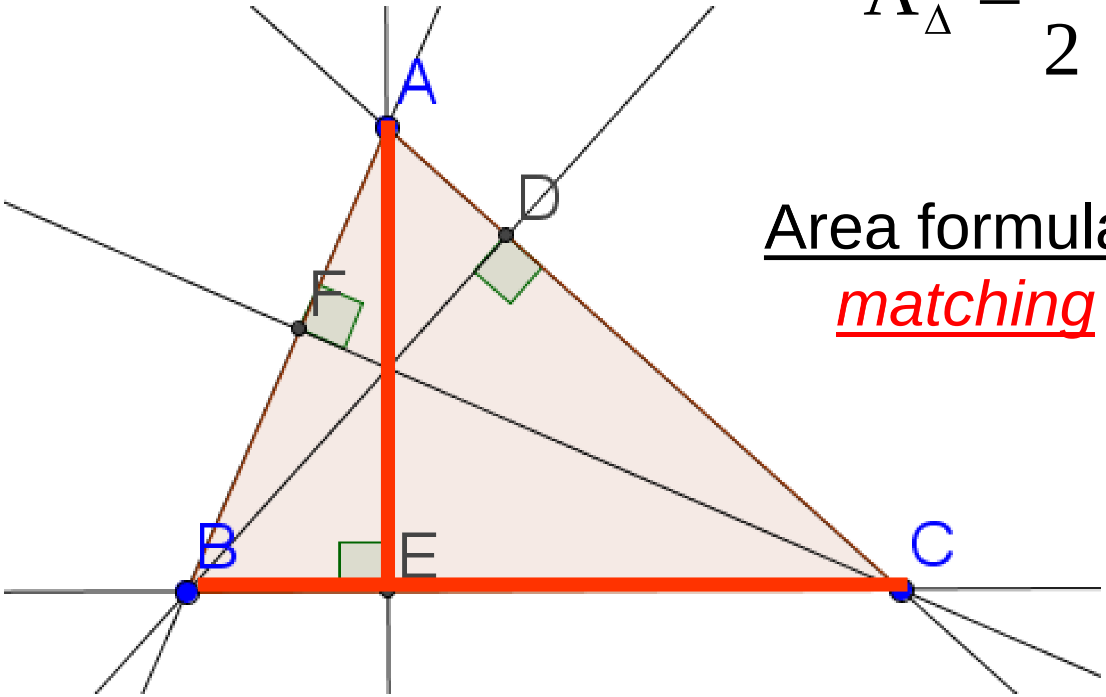
$$\boxed{A_{\Delta} = \frac{1}{2} * \text{base} * \text{height}}$$



How many different ways are there to calculate the area of a triangle?

three

$$A_{\Delta} = \frac{1}{2} * \text{base} * \text{height}$$



Area formula: requires the use of *matching* heights and sides.

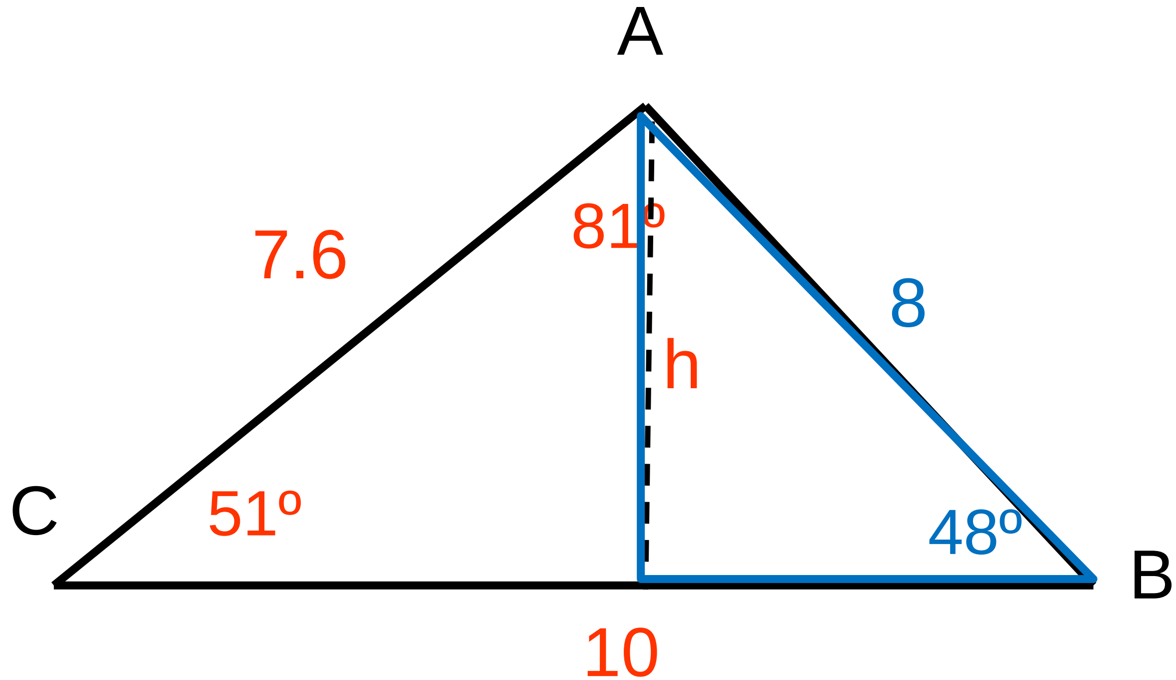
Using segment BC as the base, requires the use of segment AE as the height.

There are <u>two</u> right triangles that can be used to solve for ‘h’.

$$\sin A = \frac{opp}{hyp}$$

$$Area = 0.5 * \text{base} * \text{height}$$

$$\sin(48) = \frac{h}{8}$$



h = 8*Sin(48º)

h = 5.9

Area = ½(10)(5.9)

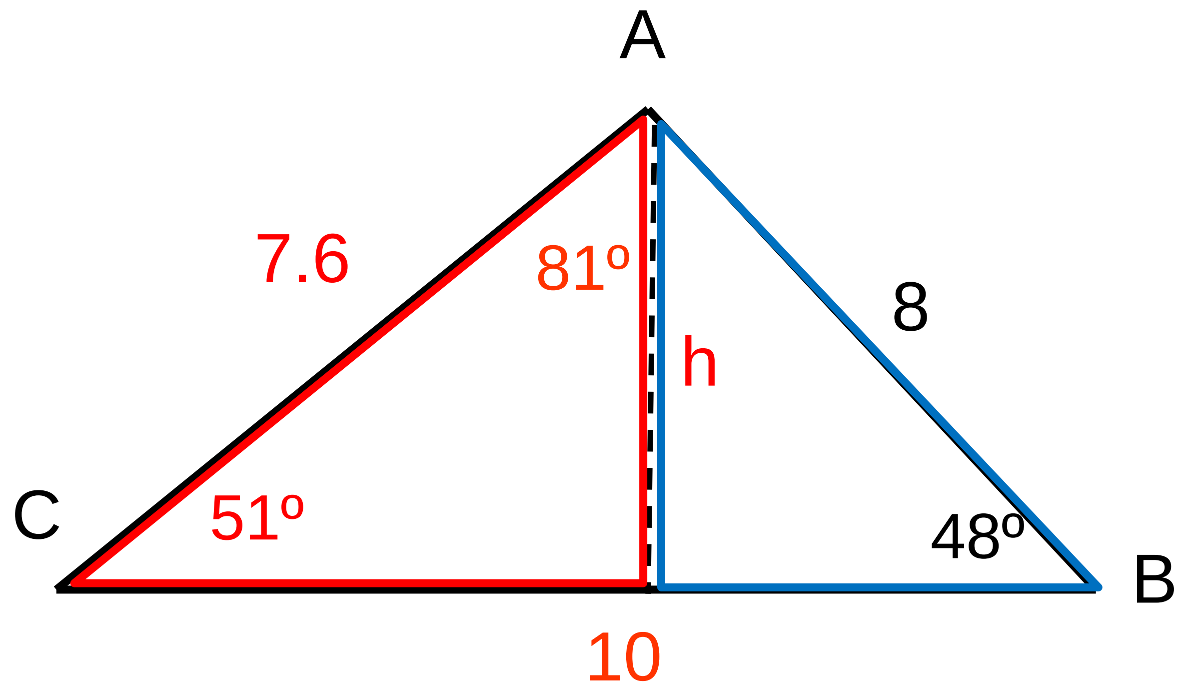
Area = 29.5 square units

For height "h" we could use the other triangle.

$$\sin A = \frac{opp}{hyp}$$

$$Area = 0.5 * \text{base} * \text{height}$$

$$\sin(51) = \frac{h}{7.6}$$



h = 7.6*Sin(51°)

h = 5.9

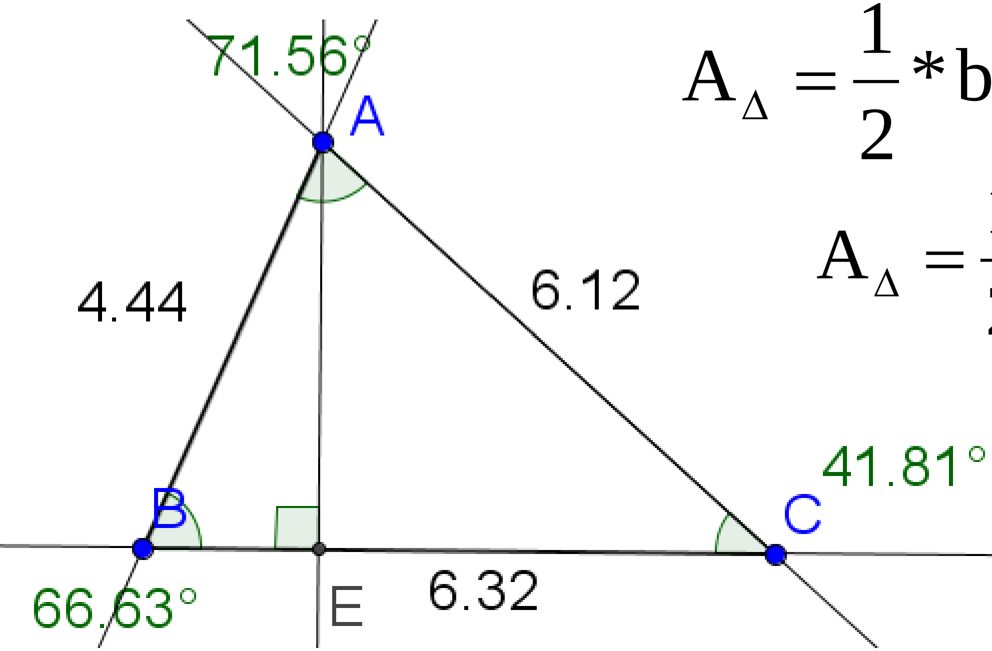
Area = ½(10)(5.9)

Area = 29.5 square units

Either of the two triangles gives us the same height.

$$AC = 6.12 \qquad AB = 4.44 \qquad BC = 6.32$$

Find the area of the triangle using altitude AE as its height.



$$\mathrm{A}_{\Delta} = \frac{1}{2} * \mathrm{base} * \mathrm{height}$$

$$\mathrm{A}_{\Delta} = \frac{1}{2} * \mathrm{BC} * \mathrm{AE}$$

$$\mathrm{A}_{\Delta} = \frac{1}{2} * (6.32) * (4.08)$$

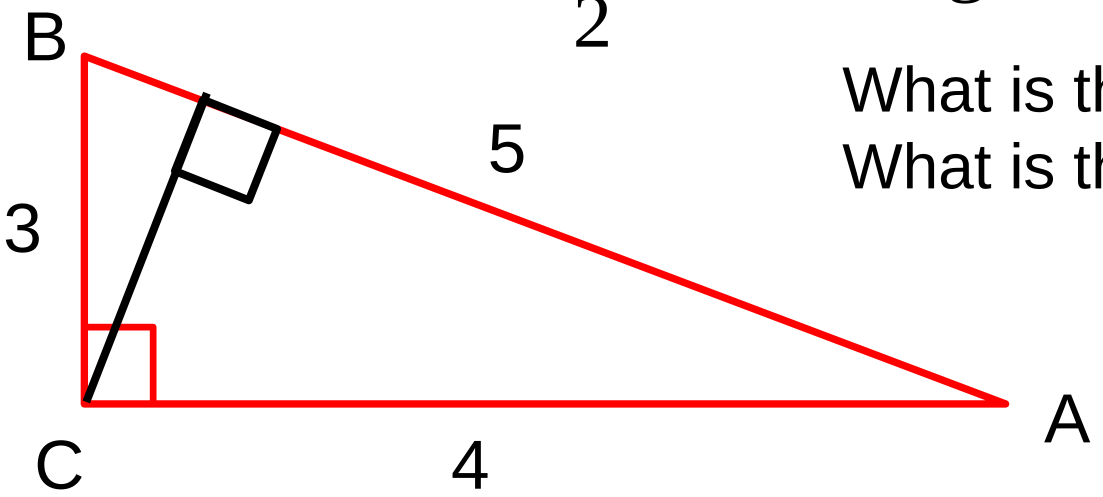
$$\boxed{\mathrm{A}_{\Delta} = 12.89 \text{ units}^2}$$

$$\boxed{\frac{\mathrm{AE}}{6.12} = \sin 41.8^{\circ}} \qquad \boxed{\frac{\mathrm{AE}}{4.44} = \sin 66.63^{\circ}}$$

$$\mathrm{A}E = 6.12 \sin 41.8^{\circ} \qquad \mathrm{A}E = 4.44 \sin 66.63^{\circ}$$

$$AE = 4.08 \qquad AE = 4.08$$

$$Area = \frac{1}{2} * \text{base} * \text{height}$$



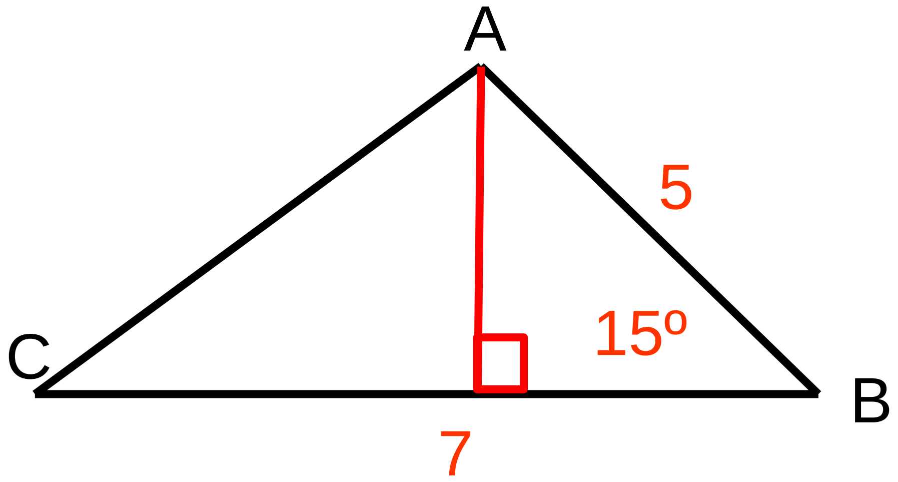
What is the base?
What is the height?

For <u>right triangles</u> one of the legs is the base, the other leg is the height (or vice-versa).

There is no point in finding the height from Angle C to side AB.

B=15, a = 7, c = 5, Find the Area of the Triangle

(Two sides and an Angle, all 3 letters are different → SAS)



$$A_{\Delta} = 0.5 * \text{base} * \text{height}$$

$$A_{\Delta} = 0.5 * (\text{Alt.}) * BC$$

$$\frac{Alt}{5} = \sin 15$$

$$Alt = 5 \sin 15$$

$$Alt = 1.29$$

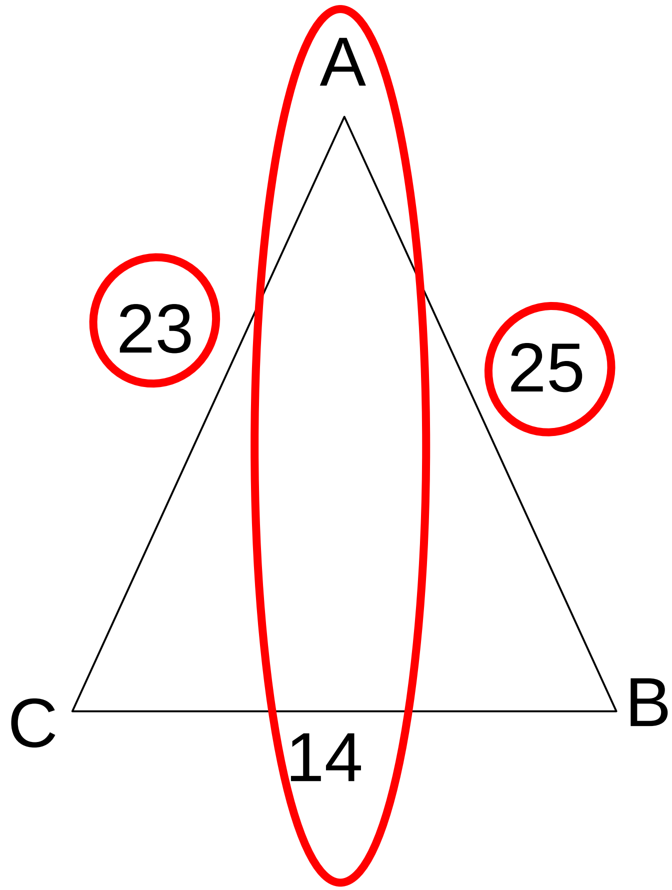
$$A_{\Delta} = 0.5 * 1.29 * 7$$

$$A_{\Delta} = 4.52\ units^2$$

What is the area of ΔABC given that $c = 25, b = 23$, and $a = 14$.

(1) Find an angle!

SSS → Law of cosines (find any angle).



$$a^2 = b^2 + c^2 - 2bc\cos A$$

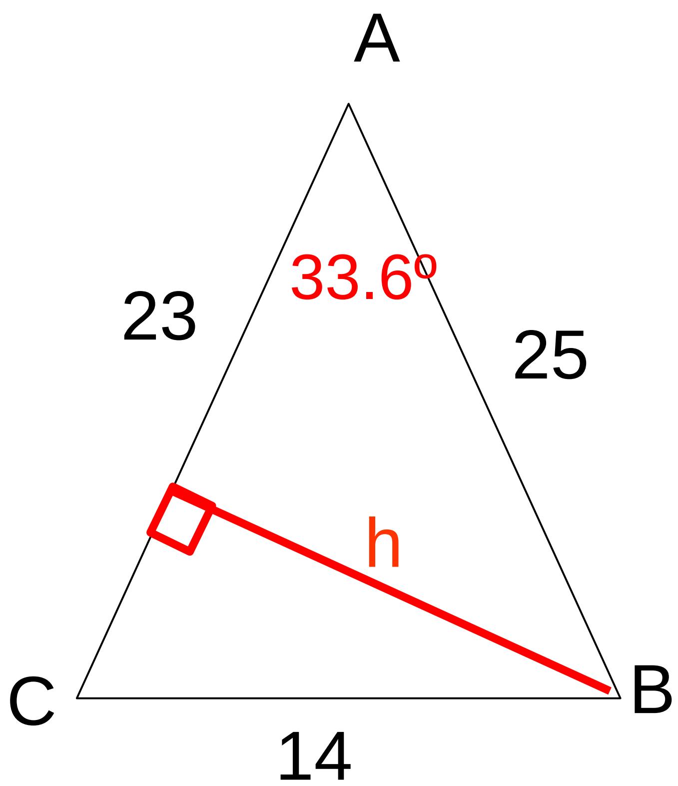
$$14^2 = 23^2 + 25^2 - 2(23)(25)\cos A$$

$$A = \cos^{-1}\left(\frac{(14^2 - 23^2 - 25^2)}{(-2 * 23 * 25)}\right)$$

$$A = 33.6^\circ$$

What is the area of ΔABC given that $c = 25$, $b = 23$, and $a = 14$.

(2) Using angle A, find the appropriate height..



$$\sin 33.6 = h/25$$

$$h = 25 \sin 33.6$$

$$h = 13.8$$

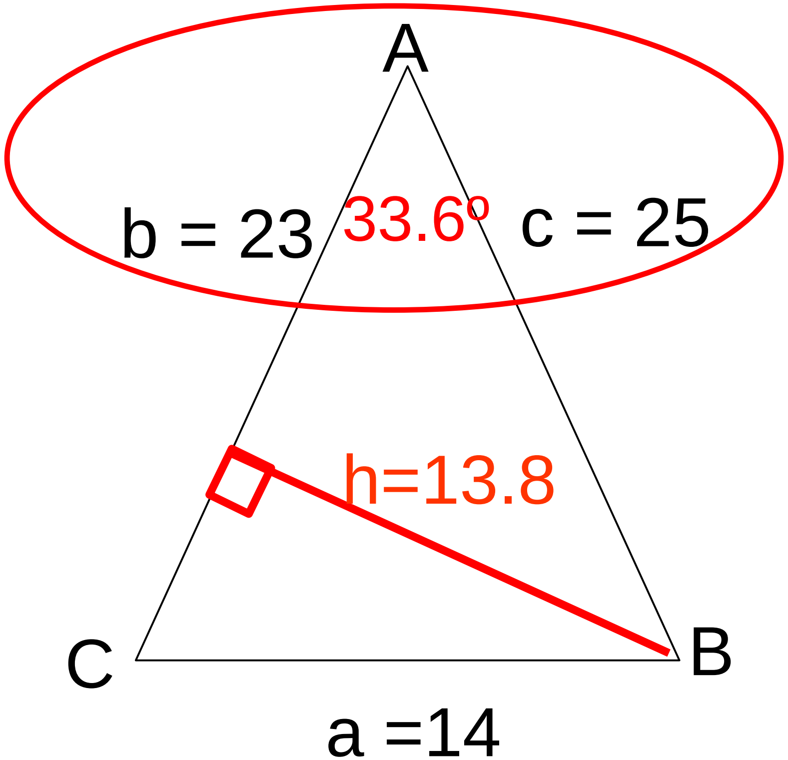
$$Area = 0.5 * \text{base} * \text{height}$$

$$Area = 0.5(23)(13.8)$$

$$\boxed{Area = 159\ units^2}$$

Let’s look at it again.

SIDE ANGLE SIDE



$$Area = 0.5 * \text{base} * \text{height}$$

$$Area = 0.5(23)(25\sin 33.6)$$

$$Area = 0.5 * b * c * \sin A$$

Notice: in the formula you see all three letters <u>‘b’, ‘c’, and ‘A’</u>

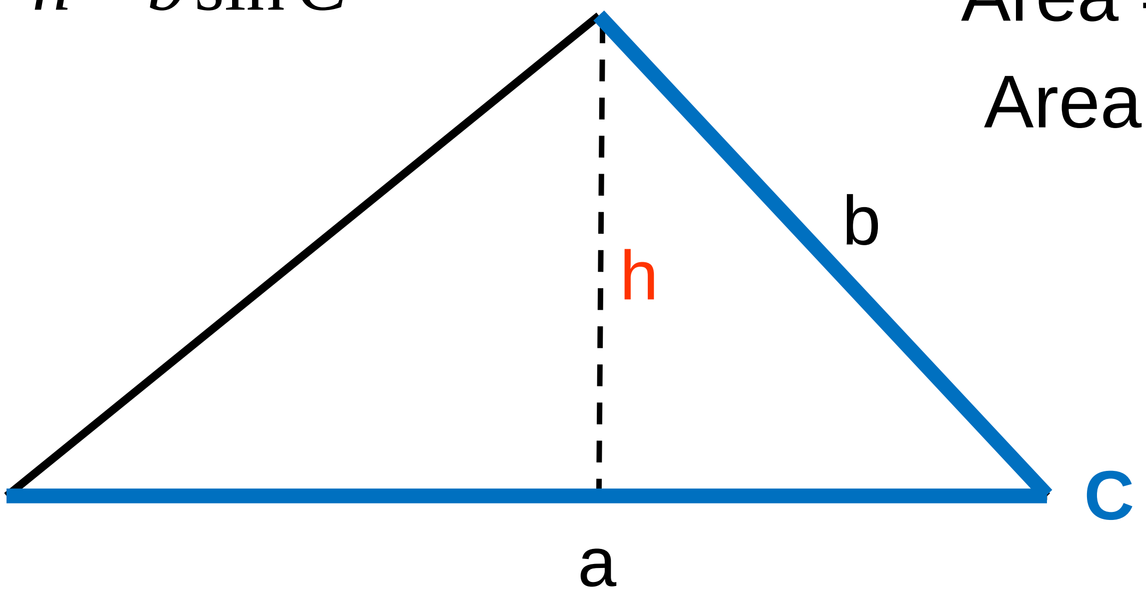
Deriving "Heron's" Formula

SIDE ANGLE SIDE

$$\sin A = \frac{opp}{hyp}$$

$$\sin C = \frac{h}{b}$$

$$h = b \sin C$$



Substitute into area formula

$$Area = \frac{1}{2} * \text{base} * \text{height}$$

Area = ½*a*b*sin C

Notice: in the formula you see all three letters 'a', 'b', and 'C'

Area = ½*b*c*sin A

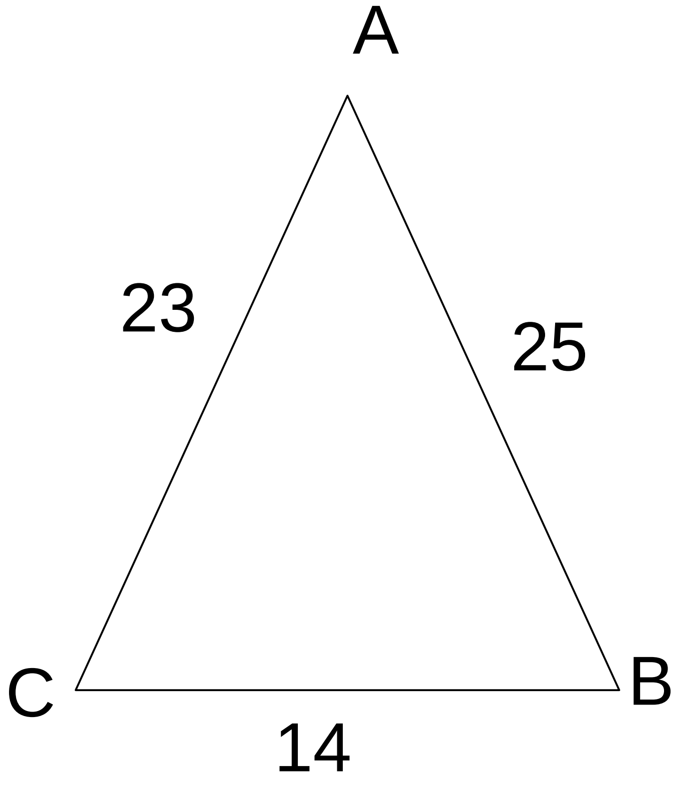
Area = ½*a*c*sin B

By combining the Law of Sines and the SAS Triangle Area formula (followed by a lot of steps), we can derive a formula for the area of a SSS triangle.

$$A = \sqrt{s(s-a)(s-b)(s-c)} \qquad s = \frac{a+b+c}{2}$$

Where 's' is the *semi-perimeter* (1/2 perimeter) of the triangle.

Find the area of the triangle.



$$s = \frac{14+23+25}{2} = 31$$

$$A = \sqrt{31(31-14)(31-23)(31-25)}$$

$$A = 159\, units^2$$