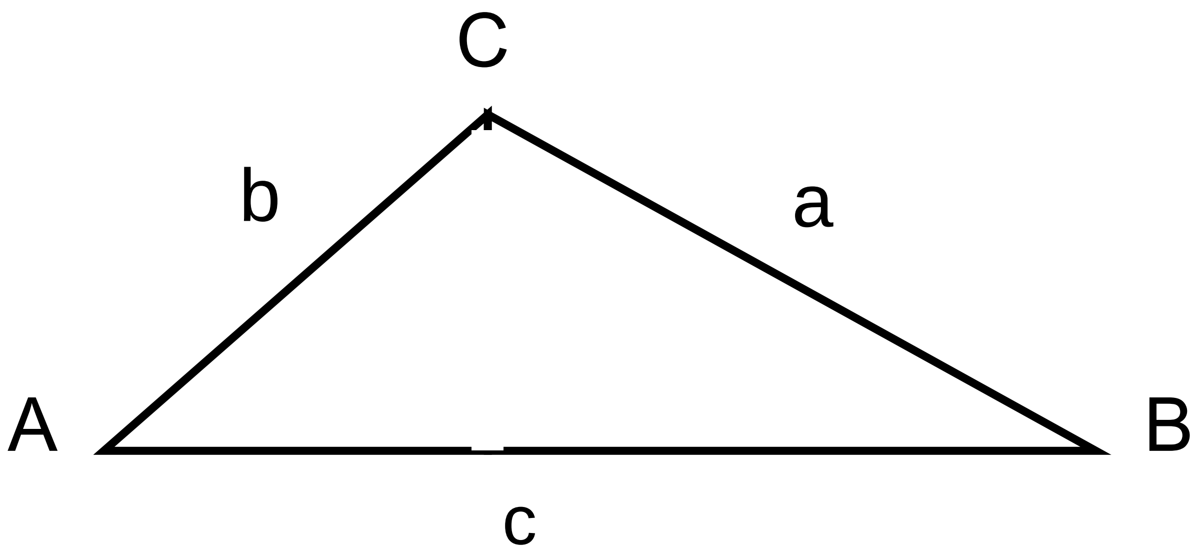
Math-1060

Lesson 4-4

The Law of Sines

Sine, cosine, and tangent ratios are for solving <u>Right triangles!</u>

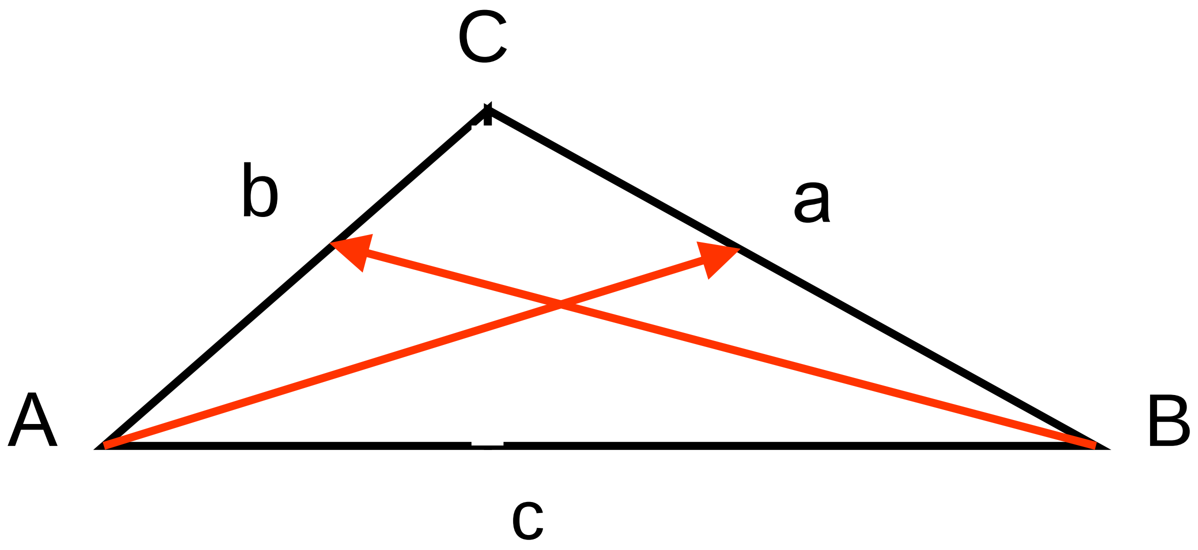
How do we solve for the unknown sides and angles of a triangle is not a right triangle?

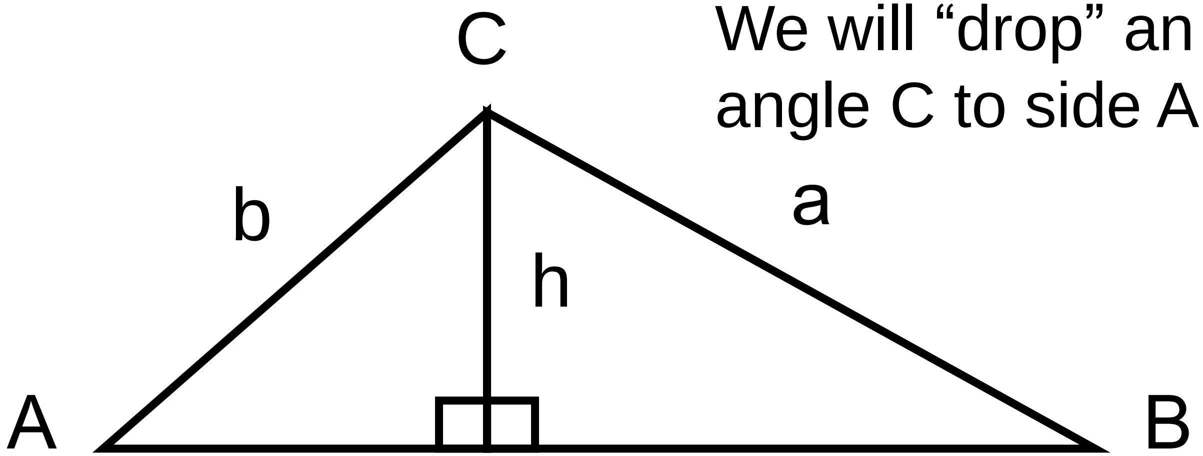


<u>The Law of Sines can be used for ALL triangles.</u>

The standard method of labeling triangles is:

The length of the side opposite Angle A is lower case a, etc.





We will "drop" an altitude from angle C to side AB.

$$\boxed{\sin A = \frac{h}{b}} \qquad \sin B = \frac{h}{a}$$

We now have <u>two right triangles</u> that we can write <u>sine ratios</u> for.

What variable is common to both equations?

$$\boxed{\begin{aligned} h &= a\sin B \\ h &= b\sin A \end{aligned}}$$

We solve each equation for 'h'.

Since 'h' = 'h', we can set the equations equal to each other (by substitution) then rearrange.

$$a\sin B = b\sin A$$

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b}}$$

We could repeat this for any combination of sides and angle.

Law of Sines

In Δ ABC with angles A, B, and C and sides a, b, and c, respectively, the following equation is true:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

By the Transitive Property this means each of the expressions are equal to each other.

We could also write it this way (using sequential property of equality steps):

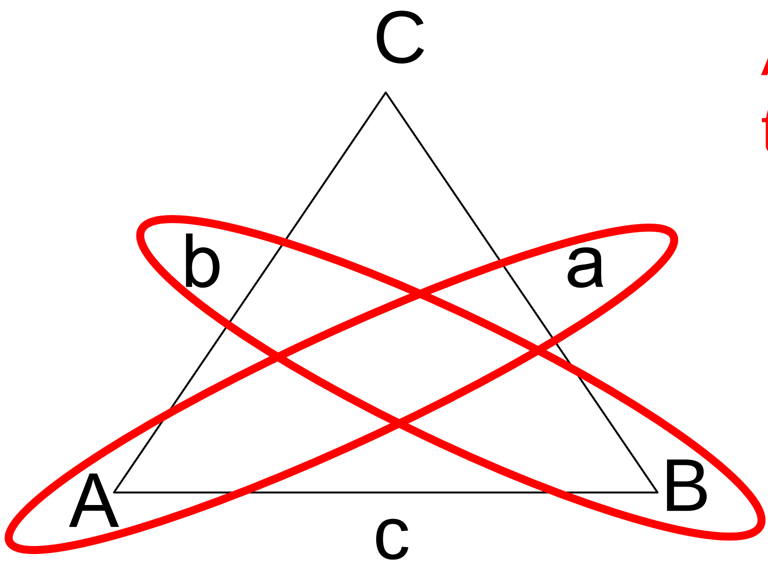
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Which one we use depends upon whether we need to find the measure of an unknown angle or an unknown side.

Pick the version that puts the unknown variable in the numerator!

There is a pattern for Law of Sines

There are six possible unknowns in a triangle (3 sides, 3 angles).



A problem will give you three of the six unknowns.

After labeling the triangle with the given information, draw the following pattern (loop the 2 sides and their opposite angles).

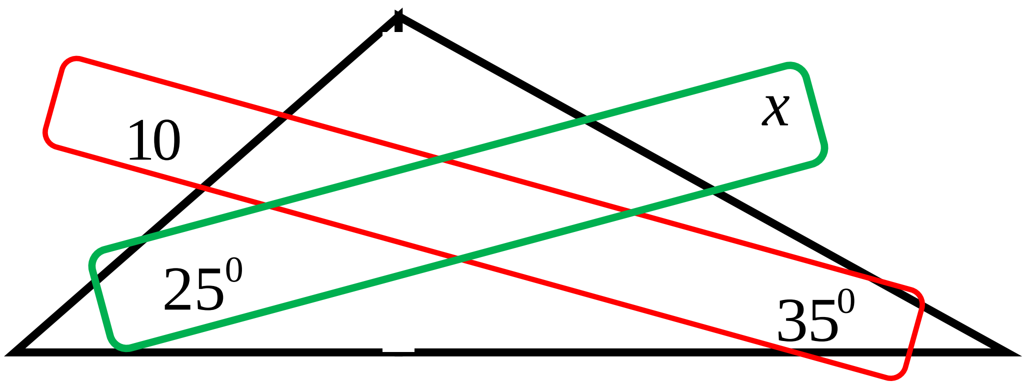
If three of the four items circled are known, use law of sines.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

Law of Sines

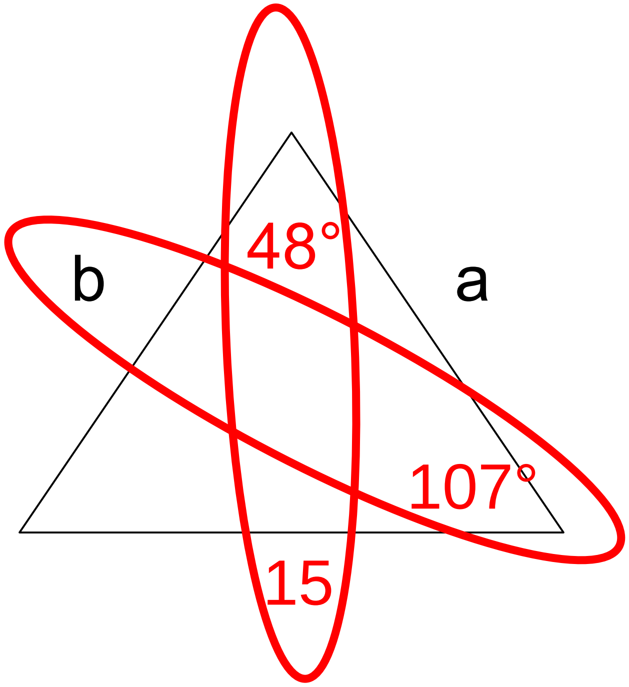
$$\frac{side_1}{\sin Angle_1} = \frac{side_2}{\sin Angle_2}$$

There is a pattern to the Law of Sines (sides and their opposite angles).



Three of the four numbers are known. You can use the proportion to solve for the unknown number.

Draw the Law of Sines Loops



You can come up with one combination of loops that gives you only <u>one unknown</u> out of 4.

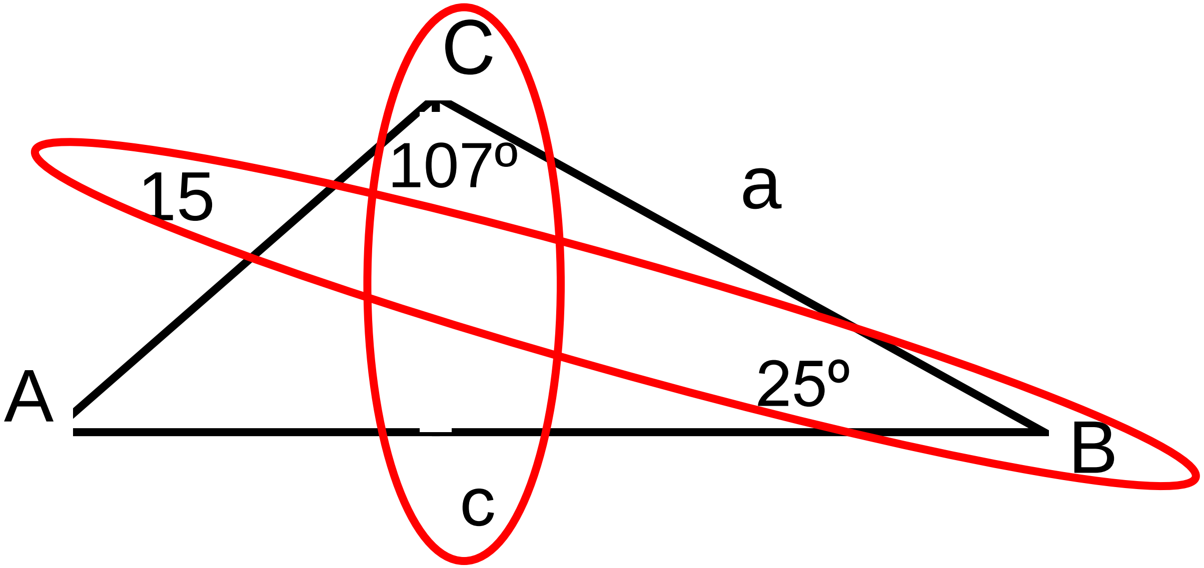
$$\frac{15}{\sin 48} = \frac{b}{\sin 107}$$

Can solve for ‘b’

Can solve for the 3rd angle

Can solve for the ‘a’

Solve for Side 'c'



1. Draw the Law of Sines Loops

2. Three of four numbers are known →

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 107^\circ} = \frac{15}{\sin 25^\circ}$$

3. Plug numbers into the Law of Sines equation and solve the unknown variable.

$$c = \frac{15 \sin 107^\circ}{\sin 25^\circ} = 33.9$$

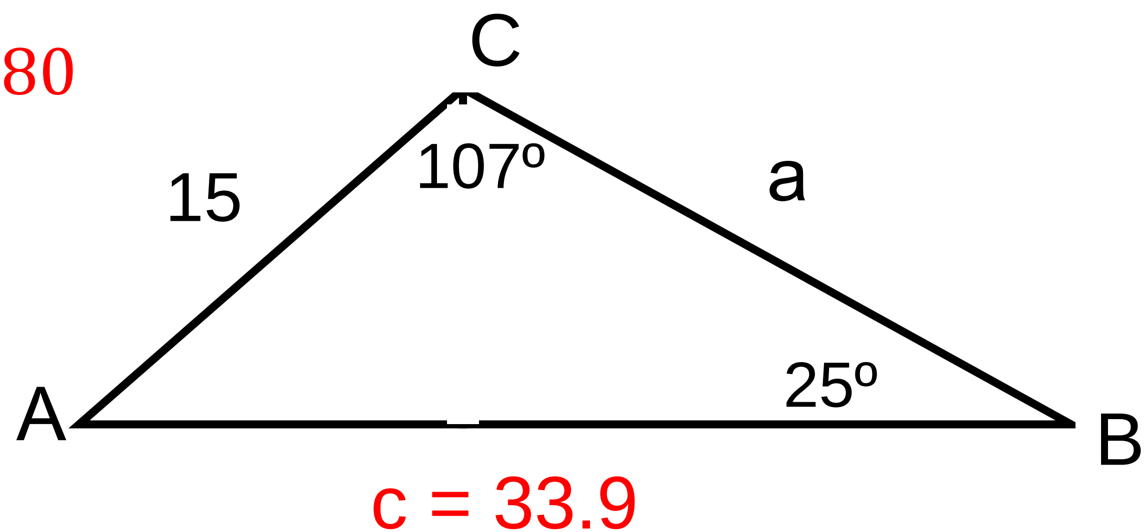
Solve for Angle A

4. All three angles of a triangle add up to 180°

$$m\angle A + m\angle B + m\angle C = 180$$

$$m\angle A + 25 + 107 = 180$$

$$m\angle A = 48$$

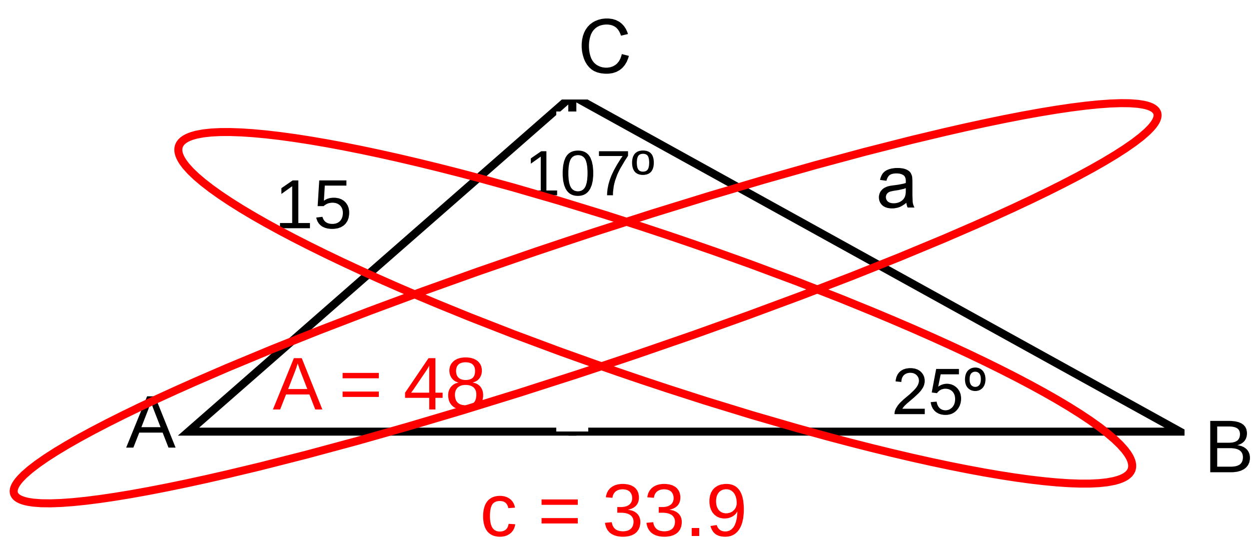


4. Solve for ‘a’
(Use Law of Sines)

$$\frac{a}{sin48} = \frac{15}{sin25}$$

$$a = \frac{15sin48}{sin25}$$

$$a = 26.4$$



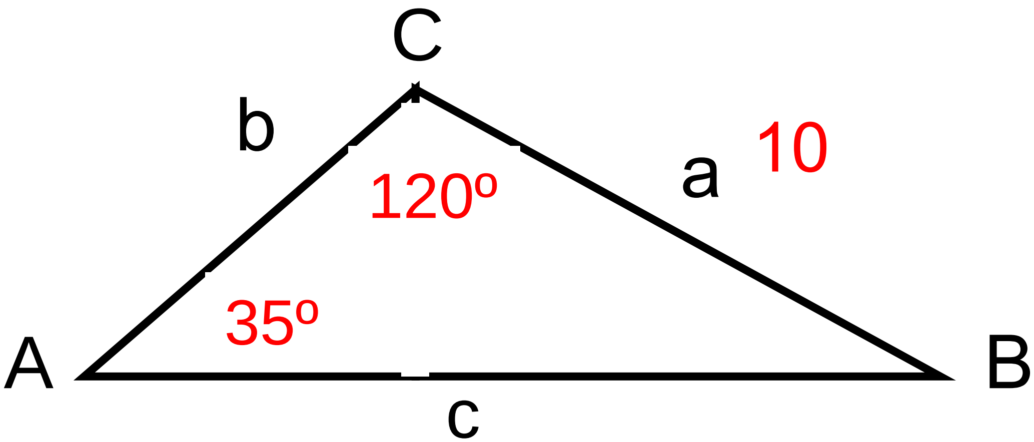
Another way a problem is given, is that they just give you some measurements.

A = 35º

a = 10

C = 120º

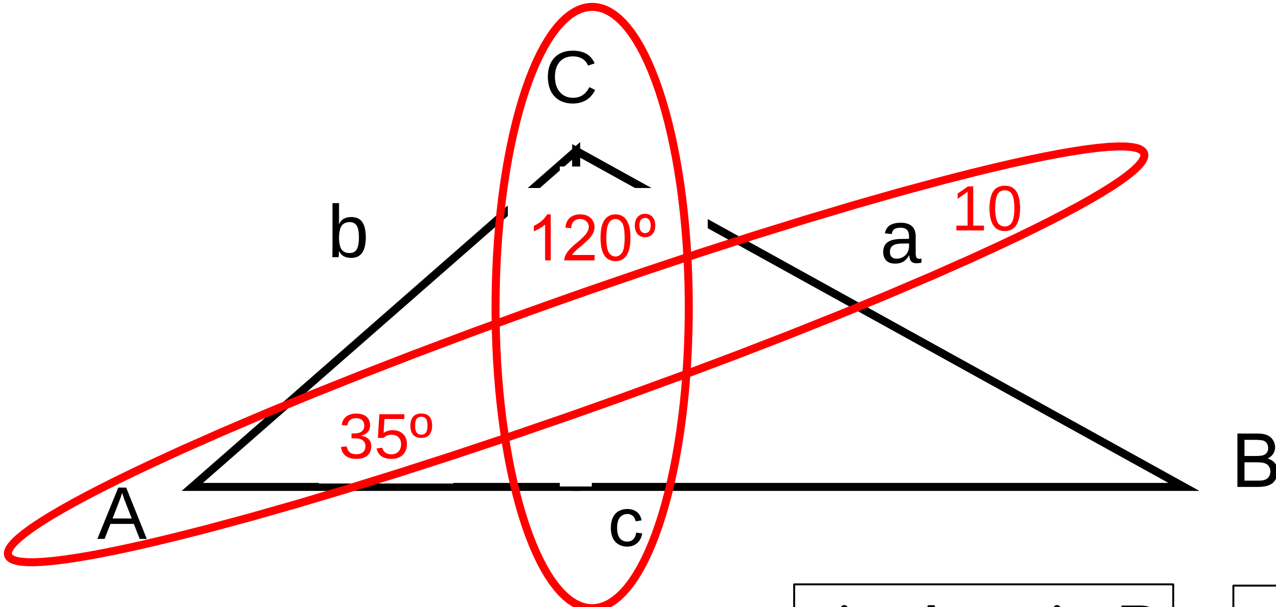
Draw the general triangle that has capital letters for angles and lower case letters for the lengths of the sides opposite the angles.



Label the triangle with values given in the problem.

A= 35º   a = 10   C = 120º

Draw the “loops” to check if Law of Sines will work.



Write the Law of Sines equation

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

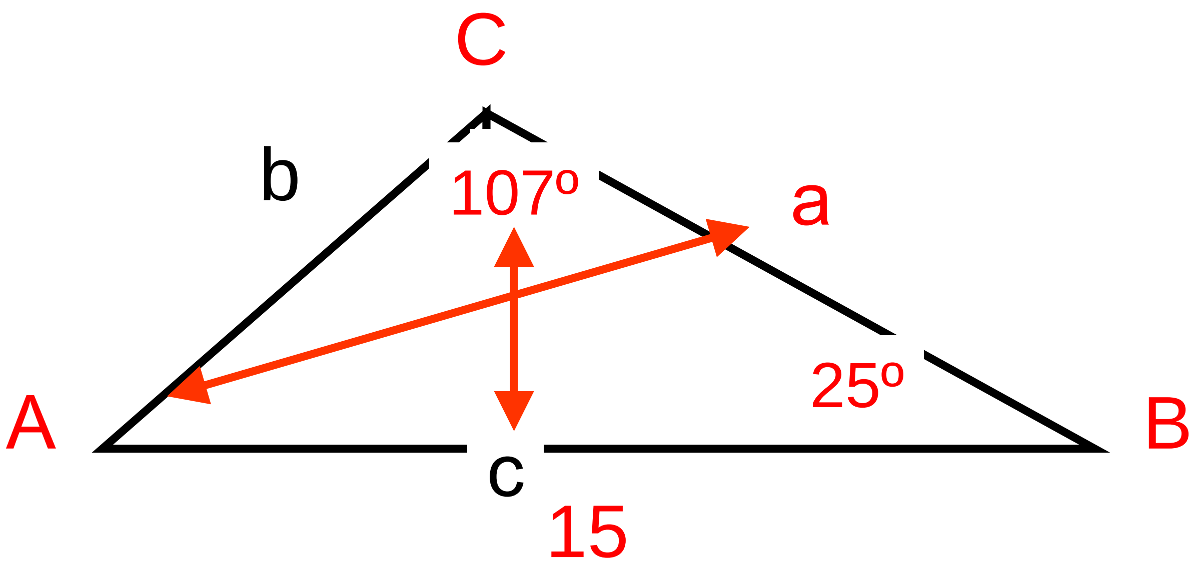
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Pick the version that <u>puts the unknown variable</u> in the <u>numerator!</u>

$$\frac{c}{\sin 120^\circ} = \frac{10}{\sin 35^\circ} \qquad c = \sin 120\left(\frac{10}{\sin 35}\right) \qquad = 15.1$$

What if they given you two angles but not the two that you need?

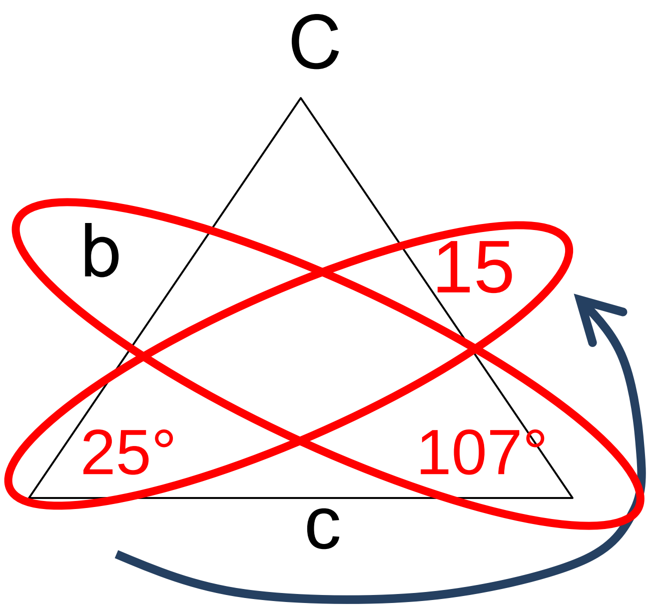
B = 25º   C = 107º   c = 15   Find "little" 'a'



Using the Triangle Sum Theorem (angle is a triangle always add up to 180º): $m\angle A = 180 - (107 + 25) = 48$

Now solve using the Law of Sines.

Triangle Review   "Walk around the block"



Start at the first side or angle that is known then list the order of the known items.

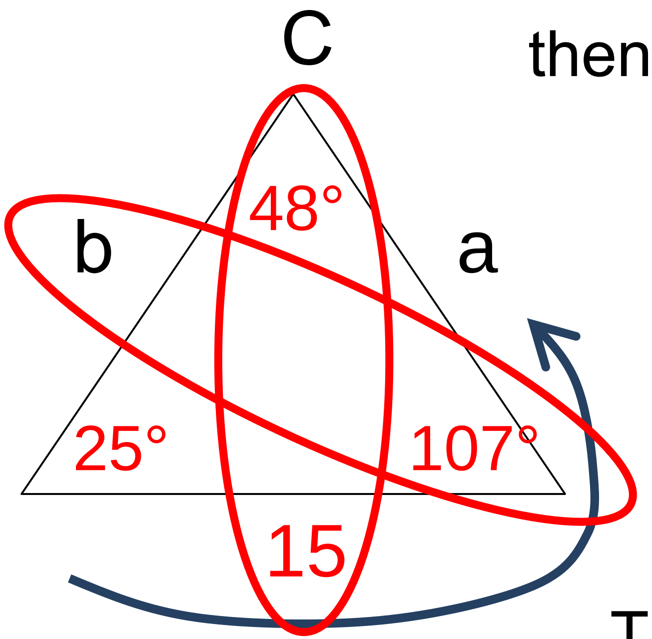
Angle, angle, side → AAS

Or, in the opposite direction:
Side, angle, angle → SAA

This means: "the measures of two angles and the non-included side are known.

Law of Sines will work for AAS or SAA.

If the following information is given "Walk around the block"



Start at the first side or angle that is known then list the order of the known items.

Angle, side, angle → ASA

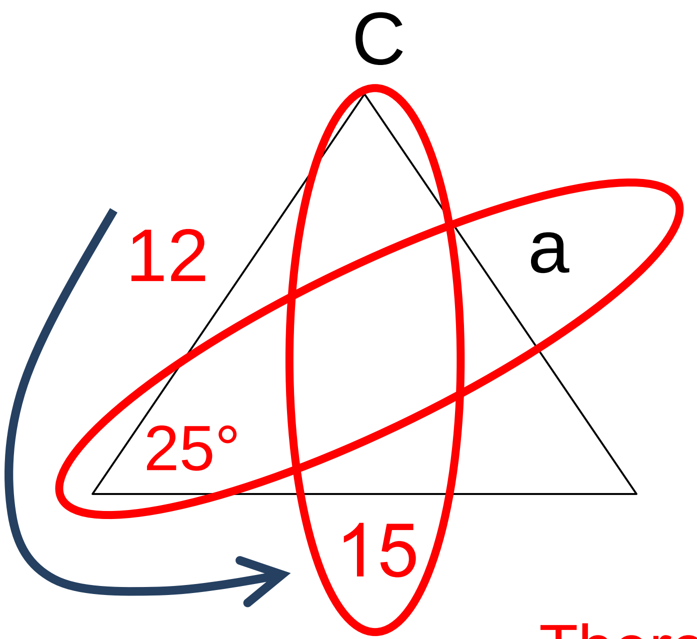
(same thing in the opposite direction: ASA)

This means: "the measures of two angles and the included side are known.

If you have two angles of a triangle, you can find the 3rd angle.

Law of Sines will work for ASA.

If the following information is given "Walk around the block"



Start at the first side or angle that is known then list the order of the known items.

Side, Angle, Side → SAS

This means: "the measures of two sides and the included angle are known.

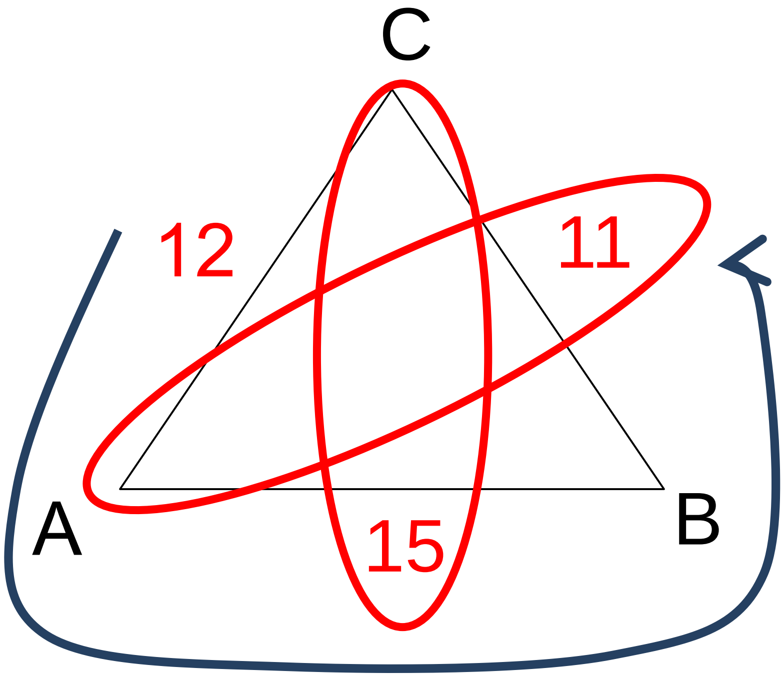
There are two unknown values in the "loops."

You cannot solve a single equation that has two unknown values!

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Law of Sines will NOT work for SAS

If the following information is given: "Walk around the block"



Start at the first side or angle that is known then list the order of the known items.

Side, Side, Side → SSS

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Law of Sines will NOT work for SSS.

Can the Law of sines be used for:

SAS ? no

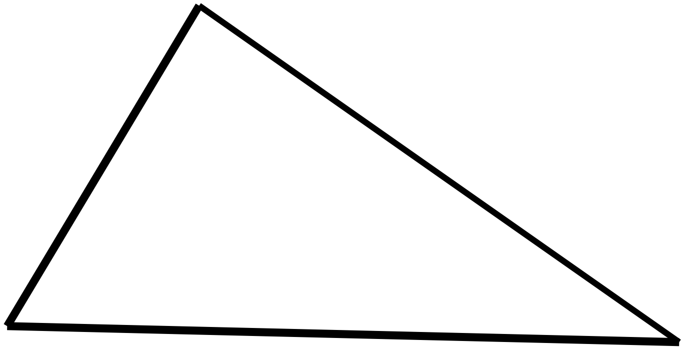
SSS ? no

SSA ? yes, BUT….

AAA? no, trangle can be scaled up or down in size (no unique triangle).

What is a triangle?

3 segments joined at their endpoints



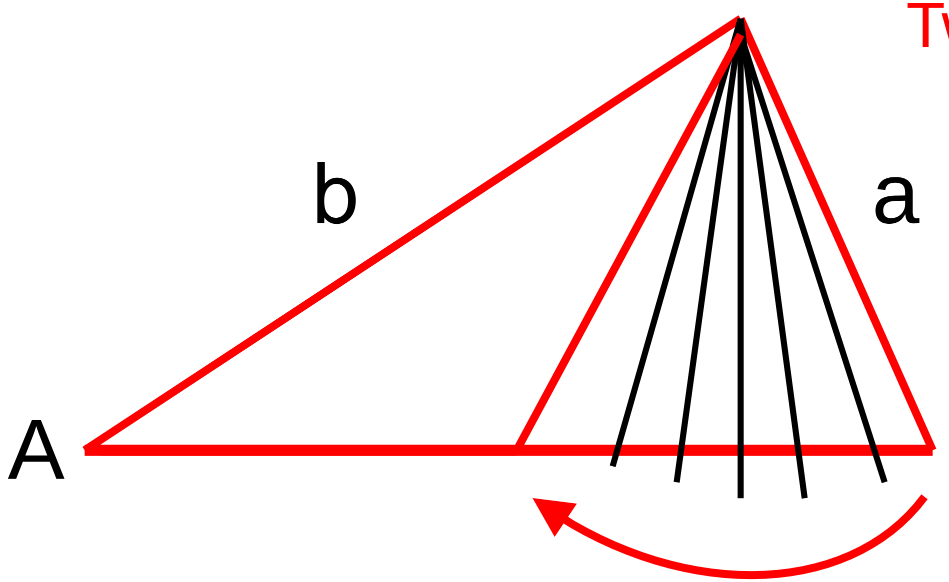
SSA: The "Ambiguous" Case

If an angle and its opposite side are known, and another side is known (Not SAS), we have a triangle.

We do not know the length of the bottom side.

We can "swing" side 'a' until it touches the bottom side at its end point. This makes another triangle.

Two triangles are possible!



SSA Case

If the given information about a triangle is:

A = 68º   b = 88   a = 85

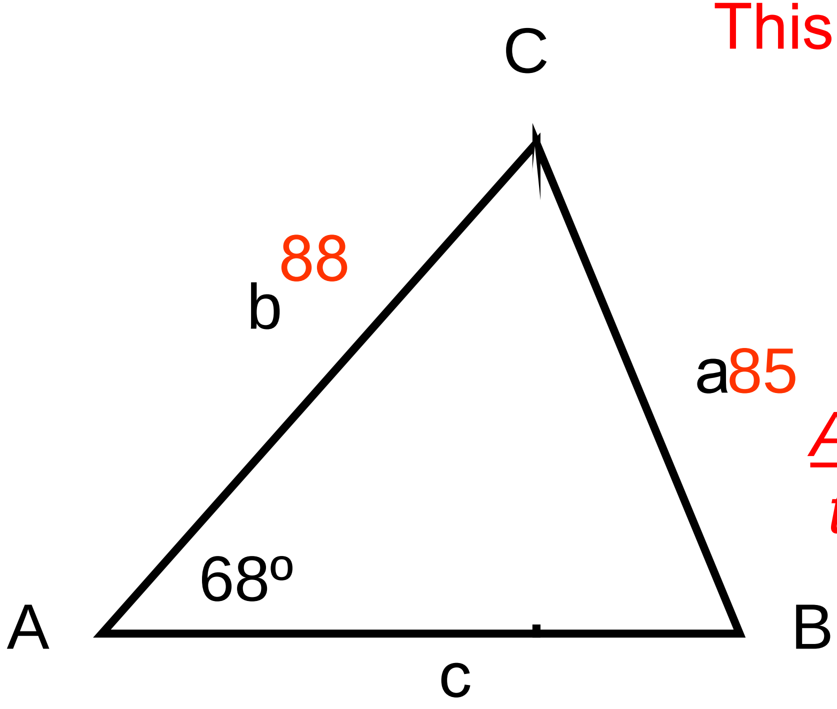
1. Draw the triangle.

2. Label the triangle

3. Determine what case it is.

The angle is not between the two sides.

This may cause problems.



We call this the *ambiguous* case.

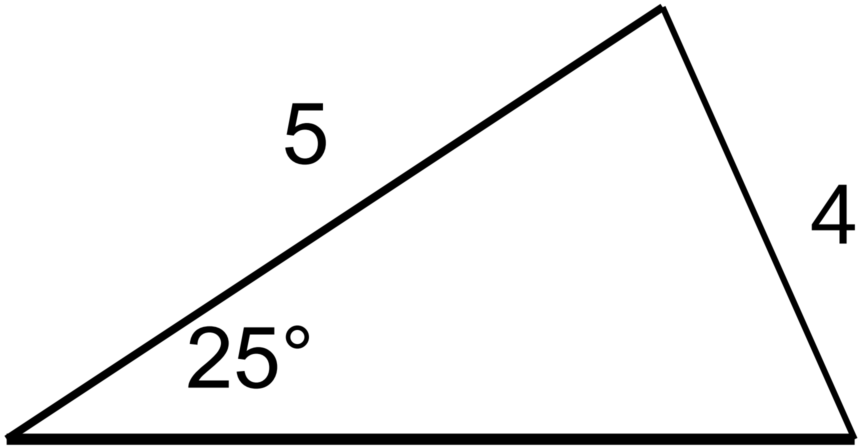
Ambiguous: Liable to more than one interpretation.

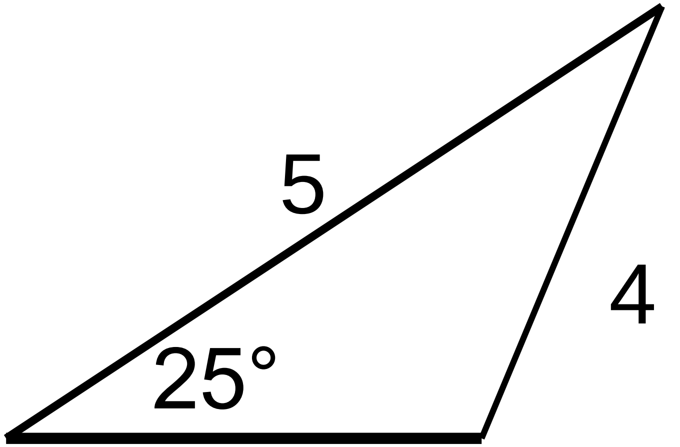
SSA: The "Ambiguous" Case

IF: (1) the side opposite the given angle is shorter than the adjacent side, and

(2) The angle is acute

→ you will have two triangles.





Which of the following cases might give you two possible triangles?

$A = 68^o$ $b = 68$ $a = 85$

$A = 25^o$ $b = 7$ $a = 5$

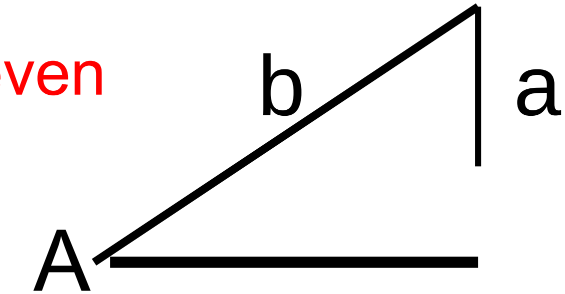
$A = 118^o$ $b = 8$ $a = 20$

Will this give you two triangles?

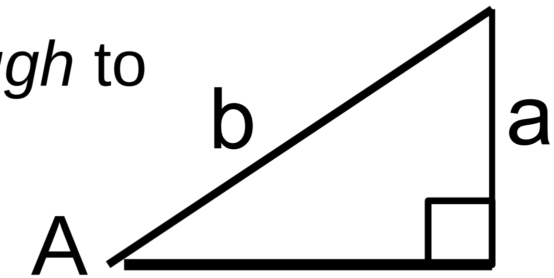
IF: The given angle is acute ($<90°$) the side opposite the given angle is shorter than the adjacent side, you may have two triangles.

IF: The given angle is acute (<90°) the side opposite the given angle is shorter than the adjacent side, there are three possibilities.

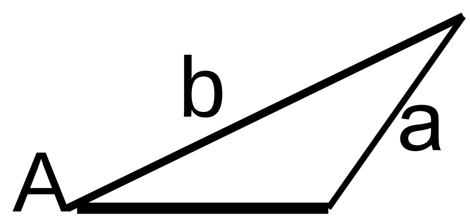
1. The opposite side is too short to even make a triangle.

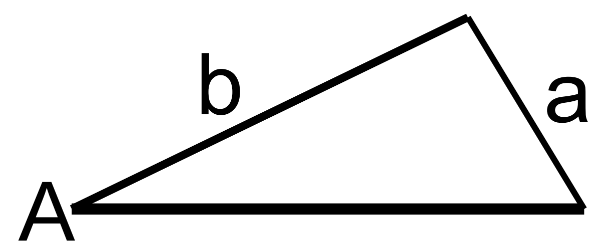


2. The opposite side is *just long enough* to touch once → right triangle.



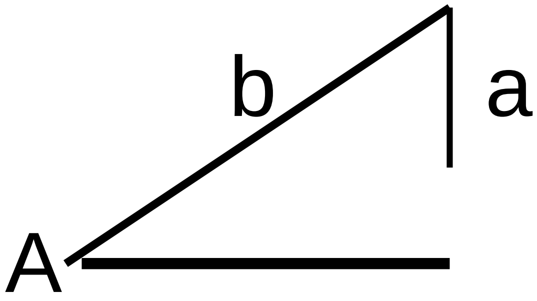
3. The opposite side can touch in two places→2 triangles.





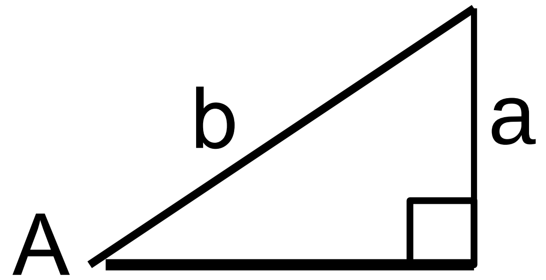
$A = 25^o$ $b = 7$ $a = 5$

How can you tell which case it is?
(0, 1, or 2 triangles)

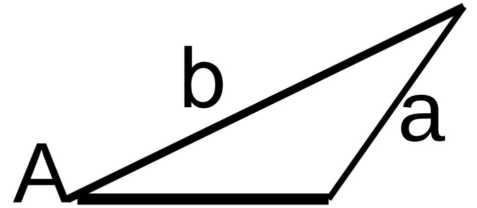


We calculate the “just right” opposite side length that will give us the right triangle.

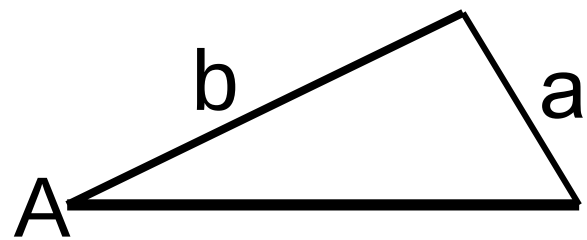
If the opposite side length is less than this “Goldilocks” length,
→ 0 triangles.



If the opposite side length is the “Goldilocks” length, it is a right triangle → 1 triangle.

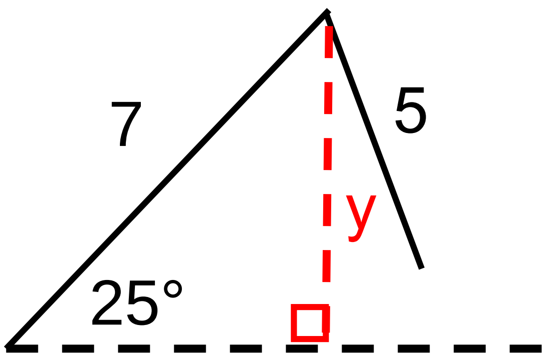


If the opposite side length is greater than this “Goldilocks” length AND is shorter than the adjacent side
→ 2 triangles.



SSA Case: Is it 0, 1, or 2 triangles?

You must calculate if the opposite side makes a right angle!



y = "Goldilocks" length

A = 25º, b = 7, a = 5

To make a right angle: 'a' = 2.95.

$$\sin 25 = \frac{y}{7}$$

y = 7 sin 25º

y = 2.95

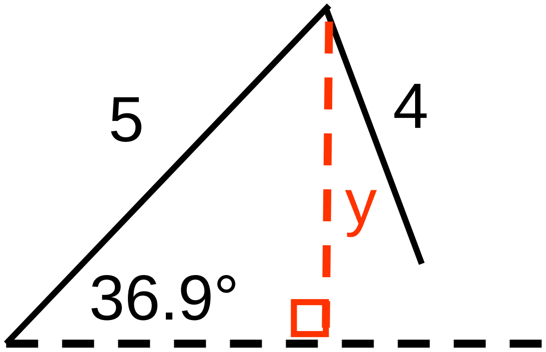
"just right" length < 'a' < adjacent side length

2.95 < (a = 5) < 7

→ 2 triangles.

A = 36.9º, b = 5, a = 4

Is this the ambiguous case?



y = length for a single, right-triangle.

3 < 4 < 5

→ 2 triangles.

$$\sin 36.9 = \frac{y}{5}$$

y = 5 sin 36.9º

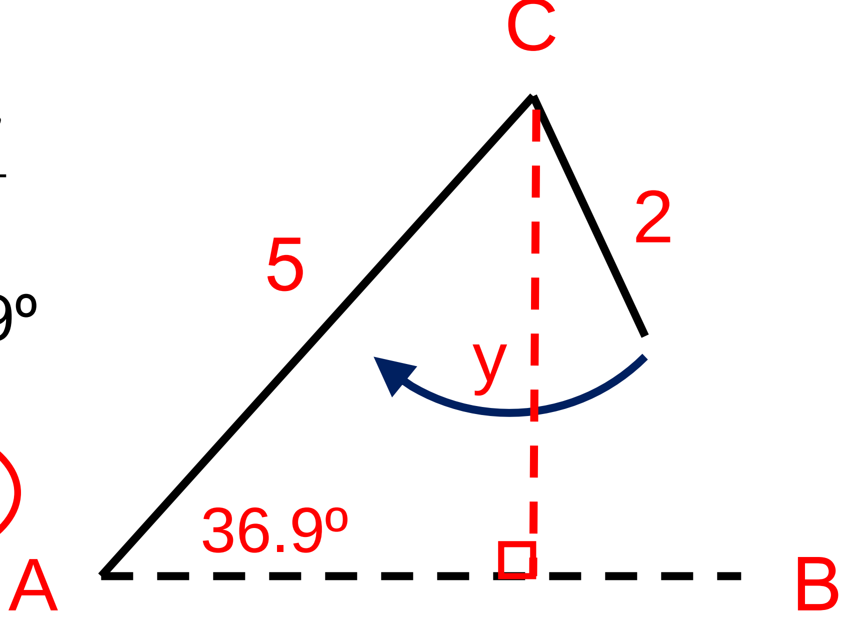
y = 3

SSA Case

$$\sin 36.9 = \frac{y}{5}$$

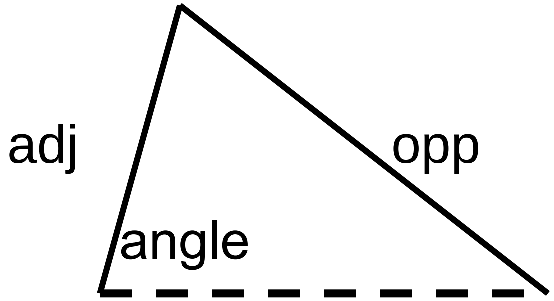
$y = 5 \sin 36.9°$

$y = 3$



Opposite side is too short → 0 triangles.

Summary (ambiguous case)



A's & S's	Given Angle and Sides	#of Triangles:
SSA	Adj > opp and opp > "just right"	2
	Adj > opp and opp = "just right"	1
	Adj > opp and opp < "just right"	0
	Adjacent side < opposite side	1

a. What is the situation? (SSA, ASA, AAS)
b. How many triangles? (one, two, or none)

$A = 52^\circ$, $a = 32$, $b = 42$

$A = 28^\circ$, $C = 75^\circ$, $c = 20$

$A = 40^\circ$, $a = 13$, $b = 16$

(Hint: draw the triangle!!!)

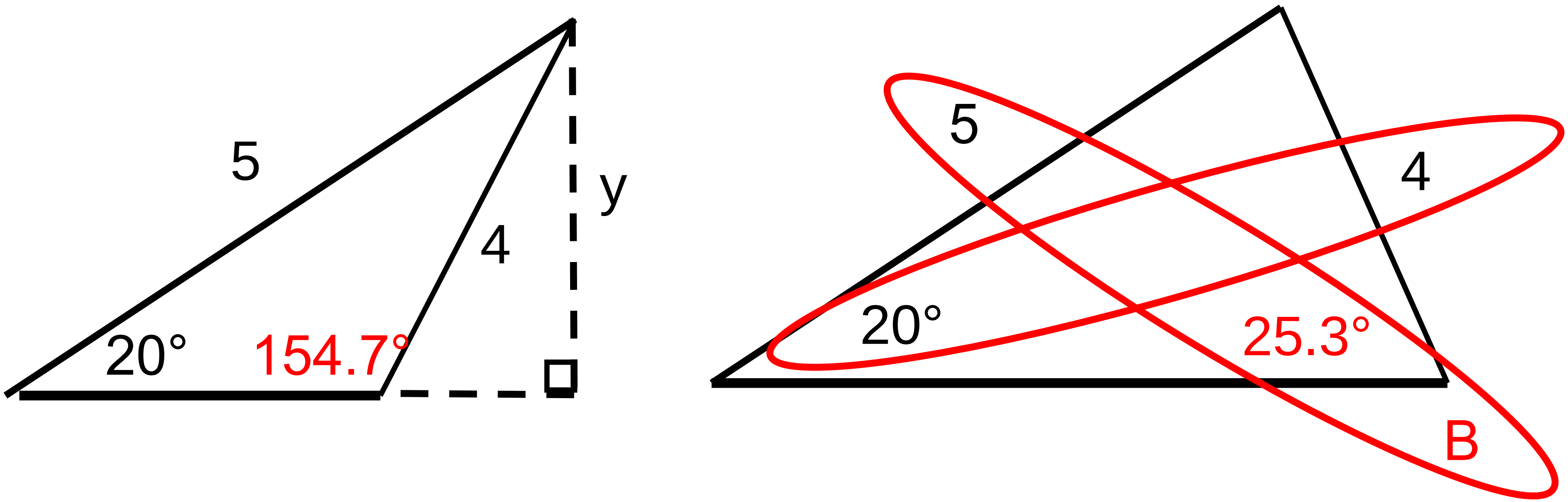
SSA: The "Ambiguous" Case A = 20º, a = 4, b = 5

$$\sin 20 = \frac{y}{5} \qquad y = 5\sin 20 \qquad = 1.71$$

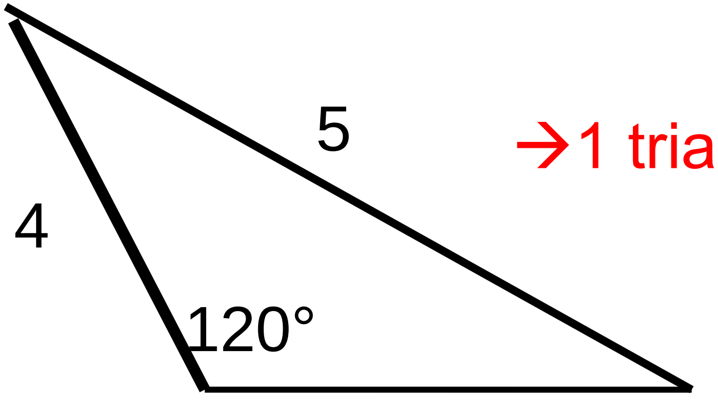
Two triangles, the law of sines will give the acute angle:

$$\frac{\sin B}{5} = \frac{\sin 20}{4} \qquad \sin B = \frac{5\sin 20}{4} \qquad \boxed{B = 25.3}$$

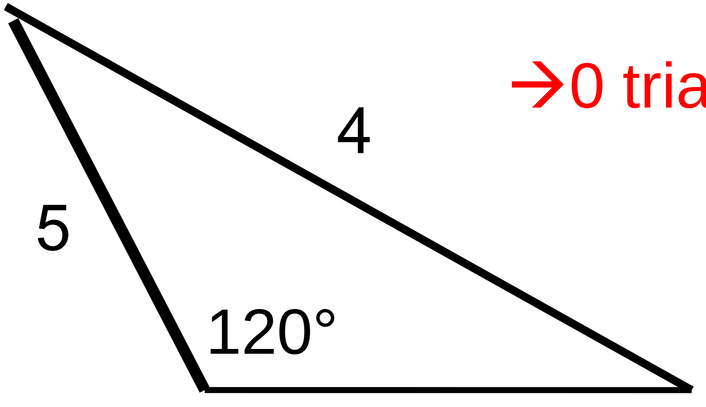
The obtuse angle is (180° - 25.3°) = 154.7°



SSA: The angle is obtuse → 0 triangles or 1 triangle.



→1 triangle

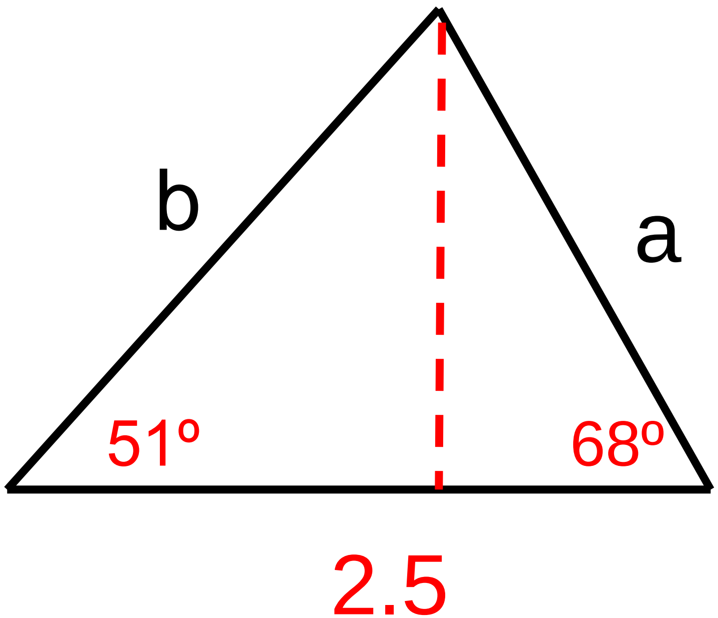


→0 triangles, why?

The longest side of a triangle *must be opposite* the largest angle.

Finding the Height of a Pole

Two people are 2.5 meters apart on opposite sides of a pole. The angles of elevation from the observers to the top of the pole are 51° and 68°. Find the height of the pole.



1. Find either 'a' or 'b' using Law of Sines.

2. Solve the right triangle using right triangle rules where <u>height</u> is the side opposite the angle.