Math-1050
Session #37

13.5: The Binomial Theorem

Powers of Binomials

- Many important mathematical discoveries have begun with the study of patterns.
- We will set the stage by observing some patterns.

Convert each binomial into a standard form polynomial

$(x+1)^0$ 1

$(x+1)^1$ $1x^1 + 1$

$(x+1)^2$ $1x^2 + 2x^1 + 1$

$(x+1)^3$ $1x^3 + 3x^2 + 3x + 1$

$(x+1)^4$ $1x^4 + 4x^3 + 6x + 4x + 1$

$(x+1)^5$ $1x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

Look for a pattern in the integers

This special arrangement of numbers is called "Pascal's Triangle."

		Exponent
$(x+1)^0$	1	0
$(x+1)^1$	1 1	1
$(x+1)^2$	1 2 1	2
$(x+1)^3$	1 3 3 1	3
$(x+1)^4$	1 4 6 4 1	4
$(x+1)^5$	1 5 10 10 5 1	5

Pascal's Triangle

This pattern was found in a Chinese text from 1303 by Chu Shih-Cheh called "diagram of the old method of finding the eighth and lower powers".

When it was introduced in Europe, Blaise Pascal used it extensively, from whom it derived its name.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Describe how the previous row's numbers can be used to predict the next row's numbers.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Now look for patterns in a generalized binomial.

$$(a+b)^0 = ? \qquad 1$$

$$(a+b)^1 = ? \qquad 1a^1 + 1b^1$$

$$(a+b)^2 = ? \qquad 1a^2 + 2a^1b^1 + 1b^2$$

$$(a+b)^3 = ? \qquad 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$(a+b)^4 = ? \qquad 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

$$(a+b)^5 = ? \qquad 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

Look for a pattern for the exponents in each row

$$(a+b)^0 = ? \qquad 1$$

$$(a+b)^1 = ? \qquad 1a^1 + 1b^1$$

$$(a+b)^2 = ? \qquad 1a^2 + 2a^1b^1 + 1b^2$$

$$(a+b)^3 = ? \qquad 1a^3 + 3a^2b^1 + 3a^1b^2 + 1b^3$$

$$(a+b)^4 = ? \qquad 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

$$(a+b)^5 = ? \qquad 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

The sum of the exponents of each term is the same as the exponent of the binomial.

$(a+b)^0 = ?$ $\qquad 1$

$(a+b)^1 = ?$ $\qquad 1a + 1b$

$(a+b)^2 = ?$ $\qquad 1a^2 + 2a^1b^1 + 1b^2$

$(a+b)^3 = ?$ $\qquad 1a^3 + 3a^2b^1 + 3a^1b^2 + 1b^3$

$(a+b)^4 = ?$ $\qquad 1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4$

In standard form, the exponents of the binomial's left side term ('a') get 1 number smaller as you move from left terms to the right.

$$(a+b)^0 = ? \qquad 1$$

$$(a+b)^1 = ? \qquad 1a + 1b$$

$$(a+b)^2 = ? \qquad 1a^2 + 2a^1b^1 + 1b^2$$

$$(a+b)^3 = ? \qquad 1a^3 + 3a^2b^1 + 3a^1b^2 + 1b^3$$

$$(a+b)^4 = ? \qquad 1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4$$

In standard form, the exponents of the binomial's right side term ('b') get 1 number smaller as you move from right terms to the left.

$(a+b)^0 = ?$ $$1$$

$(a+b)^1 = ?$ $$1a + 1b$$

$(a+b)^2 = ?$ $$1a^2 + 2ab + 1b^2$$

$(a+b)^3 = ?$ $$1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$(a+b)^4 = ?$ $$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

The number of terms in the expanded polynomial equals the exponent of the binomial plus one.

Remember synthetic division?

→ 2nd degree polynomial in standard form has 3 terms,

→ 3rd degree poly has 4 terms, etc..

We can use this pattern to find the binomial expansion of powers of binomials.

$(a+b)^0$ $\quad$ 1

$(x+1)^0$ $\quad$ 1

$(a+b)^1$ $\quad$ $1a^1 + 1b^1$

$(x+1)^1$ $\quad$ $1x^1 + 1(1)^1$

$$\boxed{x+1}$$

$(a+b)^2$ $\quad$ $1a^2 + 2a^1b^1 + 1b^2$

$(x+1)^2$ $\quad$ $1x^2 + 2x(1)^1 + (1)^2$

$$\boxed{x^2+2x+1}$$

$(a+b)^3 \qquad 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(x+1)^3 \qquad x^3 + 3x^2(1) + 3x(1)^2 + 1(1)^3$

$$\boxed{x^3 + 3x^2 + 3x + 1}$$

$(a+b)^4 \qquad 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

$(x+1)^4 \qquad 1x^4 + 4x^3(1) + 6x^2(1)^2 + 4x(1)^3 + 1(1)^4$

$$\boxed{x^4 + 4x^3 + 6x^2 + 4x + 1}$$

“Order Matters” vs. “Order Doesn’t Matter”

Different order → separate items "Permutation"

“ ‘n’ items taken ‘r’ at a time”

$${}^{n}_{r}P = \frac{n!}{(n-r)!}$$

Different Order → not separate items "Combination"

must divide out the double counting

“ ‘n’ choose ‘r’ items”

$${}^{n}_{r}C = \frac{n!}{r!\,(n-r)!}$$

Our book uses the notation: $\binom{n}{r}$

$${}^{0}_{0}C = \frac{0!}{0!\,(0-0)!} = 1 \qquad 0! = 1 \qquad {}^{n}_{r}C = \frac{n!}{r!\,(n-r)!}$$

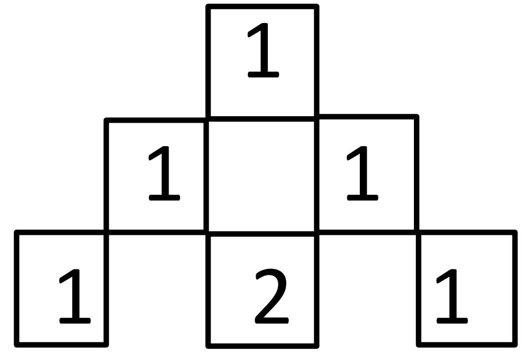
$${}^{1}_{0}C = \frac{1!}{0!\,(1-0)!} = 1$$

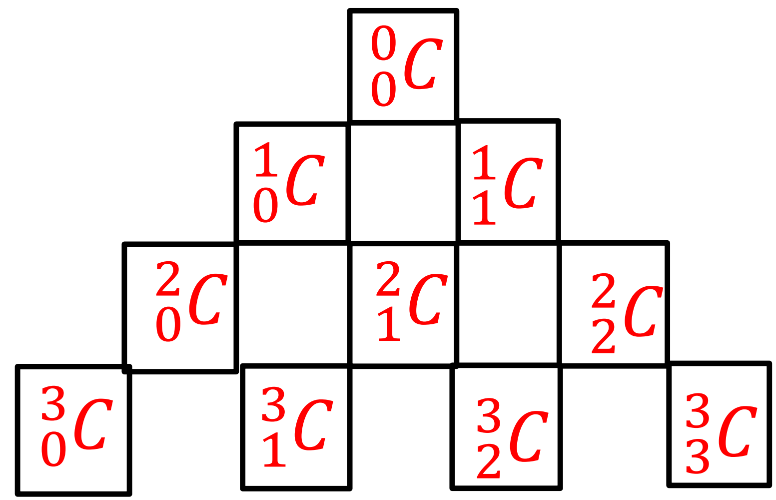
$${}^{1}_{1}C = \frac{1!}{1!\,(1-1)!} = 1$$

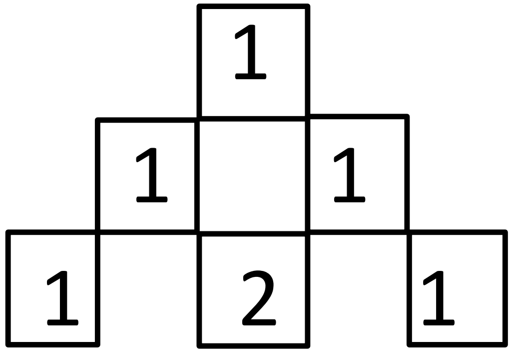
$${}^{2}_{0}C = \frac{2!}{0!\,(2-0)!} = 1$$

$${}^{2}_{1}C = \frac{2!}{1!\,(2-1)!} = 2$$

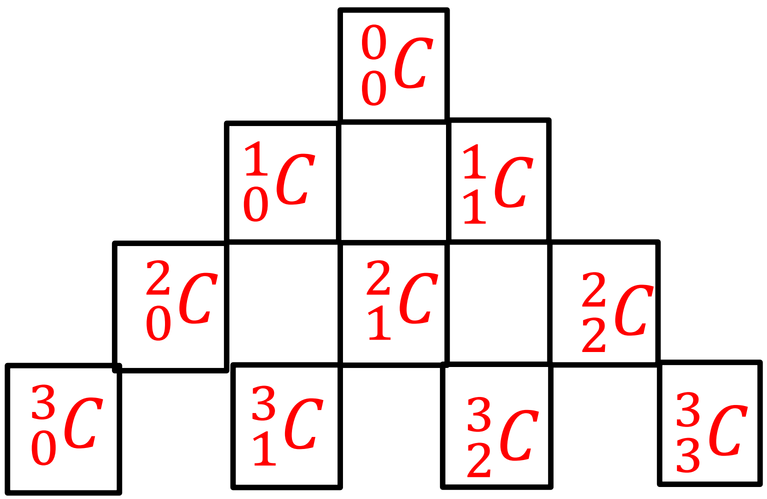
$${}^{2}_{2}C = \frac{2!}{2!\,(2-2)!} = 1$$

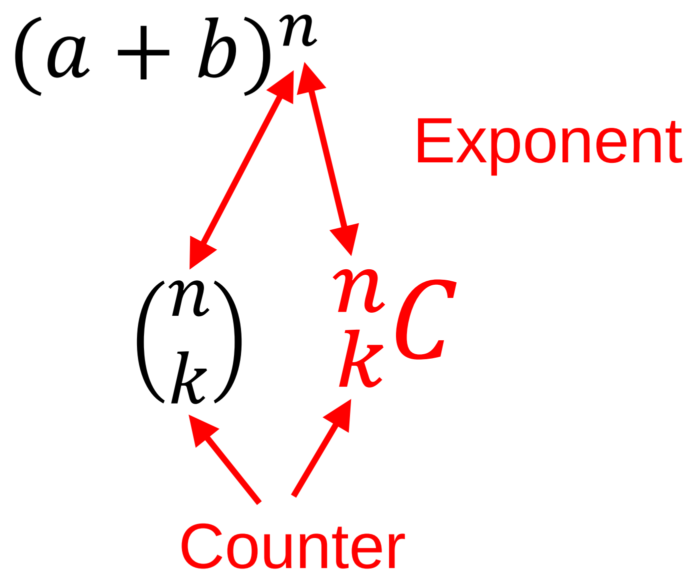






$${}^{n}_{r}C = \frac{n!}{r!\,(n-r)!}$$





Instead of memorizing the pattern of the binomial expansion (which is not very difficult), we will use the ***BINOMIAL THEOREM***.

Binomial Theorem: If a binomial has an exponent, then it can be converted to a standard form polynomial using the formula:

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k x^{n-k} =$$

$$\binom{n}{0} x^n + \binom{n}{1} a x^{n-1} + \binom{n}{2} a^2 x^{n-2} + \cdots + \binom{n}{k} a^k x^{n-k} + \cdots \binom{n}{n} a^n$$

$$(x+3)^4 = \binom{4}{0} x^4 + \binom{4}{1} 3^1 x^3 + \binom{4}{2} 3^2 x^2 + \binom{4}{3} 3^3 x^1 + \binom{4}{4} 3^4$$

$$1x^4 \quad +4(3)x^3 \quad +6(3^2)x^2 \quad +4(3^3)x \quad +3^4$$

$$(x+3)^4 = x^4 + 12x^3 + 54x^2 + 108x + 81$$

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k x^{n-k}$$

$$(2x-5)^4 = ?$$

$$(3x-5)^4 = \binom{4}{0}(3x)^4 \qquad = 81x^4$$

$$+\binom{4}{1}(-5)(3x)^3 \qquad -540x^3$$

$$+\binom{4}{2}(-5)^2(3x)^2 \qquad +1350x^2$$

$$+\binom{4}{3}(-5)^3(3x)^1 \qquad -1500x$$

$$+\binom{4}{4}(-5)^4 \qquad +625$$

$$(3x-5)^4 = 81x^4 - 540x^3 + 1350x^2 - 1500x + 625$$

Finding a specific coefficient in a binomial expansion

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k x^{n-k}$$

$$\binom{n}{0} x^n + \binom{n}{1} a x^{n-1} + \binom{n}{2} a^2 x^{n-2} + \cdots + \binom{n}{k} a^k x^{n-k} + \cdots \binom{n}{n} a^n$$

Find the coefficient of x^7 in the expansion of $(2x-4)^{11}$

$$\binom{11}{0}(2x)^{11} + \binom{11}{1}(-4)(2x)^{10} + \binom{11}{2}(-4)^2 x^9$$

$$+ \binom{11}{3}(-4)^3(2x)^8 + \cdots$$

$-2{,}703{,}360$

Write a generalized formula to find the coefficient of the term with an exponent of "*j*" in the binomial expansion of $(ax+b)^n$

$$\binom{n}{n-j}(b)^{n-j}(a^j)x^j$$

Finding a specific term in a binomial expansion

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k x^{n-k}$$

$$\binom{n}{0} x^n + \binom{n}{1} a x^{n-1} + \binom{n}{2} a^2 x^{n-2} + \cdots + \binom{n}{k} a^k x^{n-k} + \cdots \binom{n}{n} a^n$$

Find the 4th term in the expansion of $(3x-2)^9$

$$\binom{9}{0}(3x)^9 + \binom{9}{1}(-2)(3x)^8 + \binom{9}{2}(-2)^2(3x)^7$$

$$+ \binom{9}{3}(-2)^3(3x)^6 + \cdots$$

$$-489{,}888x^6$$

Write a generalized formula to find the "*j-th*" term in the binomial expansion of $(ax+b)^n$

$$\binom{n}{j-1}(b)^{j-1}(a^{k-j+1})x^{k-j+1}$$

The Fibonacci Sequence

The Fibonacci sequence can be defined recursively by

$a_1 = 1$

$a_2 = 1$

$a_n = a_{n-2} + a_{n-1}$

for all positive integers $n \geq 3$.

$$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$