Math-1050
Session #35

13.3: Geometric Sequence

Arithmetic Sequence: each previous term has a number added to it to get the next number in the sequence..

n	1	2	3	4
a_n	1	4	7	10

Recursive formula: $a_1 = 1$ $a_n = a_{n-1} + 3$ For: n = 2, 3, 4

Explicit formula: $a_n = 3n - 2$ For: n = 1, 2, 3, 4

$$a_n = 3(n-1) + 1$$

Geometric Sequence: each previous term is multiplied by a number to get the next number in the sequence.

$$a_1 = 6 \qquad a_n = 2a_{n-1}$$ For: n = 2, 3, 4

The "input" is a subset of the natural #'s

n	1	2	3	4
a_n	6	12	24	48

"The output doubles (2 times as big)"

$$\frac{a_2}{a_1} = \frac{12}{6} = 2 \qquad \frac{a_3}{a_2} = \frac{24}{12} = 2$$

We call '2' the "common ratio"→ "r"

$$r = \frac{a_n}{a_{n-1}} = 2$$

recursively defined geometric sequence

n	1	2	3	4
a_n	6	12	24	48

Define 1st term

Define "next" term

Define "counter" (finite or infinite sequence)

$$a_1 = (some\#) \qquad a_n = r * a_{n-1}$$

For: n = 2, 3, 4

$$a_1 = 6 \qquad a_n = 2a_{n-1}$$

For: n = 2, 3, 4

Where "r" is the common ratio.

Explicitly defined geometric sequence

n	1	2	3	4
a_n	6	12	24	48

Define "counter" (finite or infinite sequence)

$$a_n = a_1 * r^{n-1}$$

For: n = 1, 2, 3, 4

Where "r" is the common ratio.

$$a_n = 6(2)^{n-1}$$

For: n = 1, 2, 3, 4

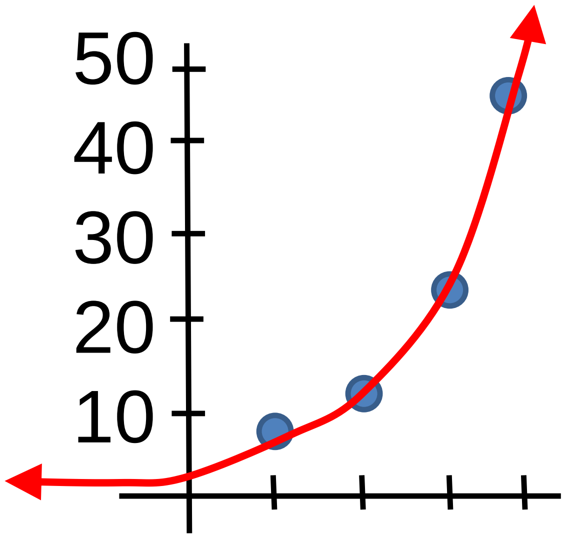
Why use the exponent "n – 1" in this formula?

Zero Exponent Property → so that the coefficient will be the 1st term

The geometric sequence is similar to which function?

$$a_n = 6(2)^{n-1}$$

$$a_1 = 6 \qquad a_n = 2a_{n-1}$$



n	**0**	1	2	3	4
a_n	**3**	6	12	24	48

Exponential Function

$$f(x) = ab^x$$

Remember "initial value"? **"growth factor"?**

initial value: the coefficient of the exponential function.

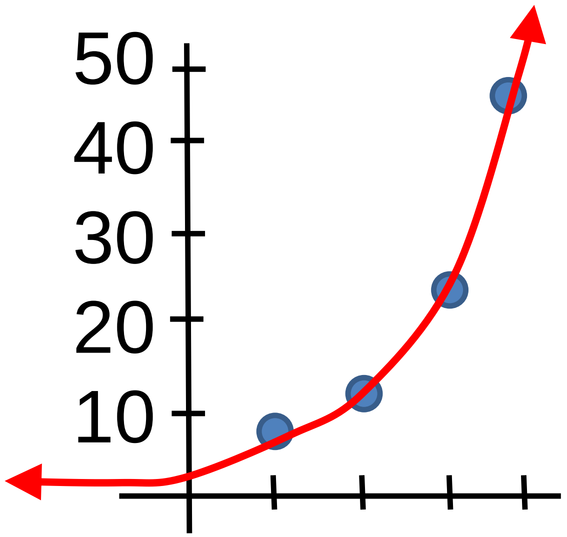
$$f(0) = ? \qquad a = 3 \qquad f(x) = 3 * b^x$$

$$f(x) = 3(2)^x$$

How is geometric sequence similar to the Exponential function?

$$f(x) = 3(2)^x$$

$$a_n = 6(2)^{n-1}$$ For: n = 1, 2, 3, 4



n	0	1	2	3	4
a_n	3	6	12	24	48

Similar:

"common ratio" of geometric sequence

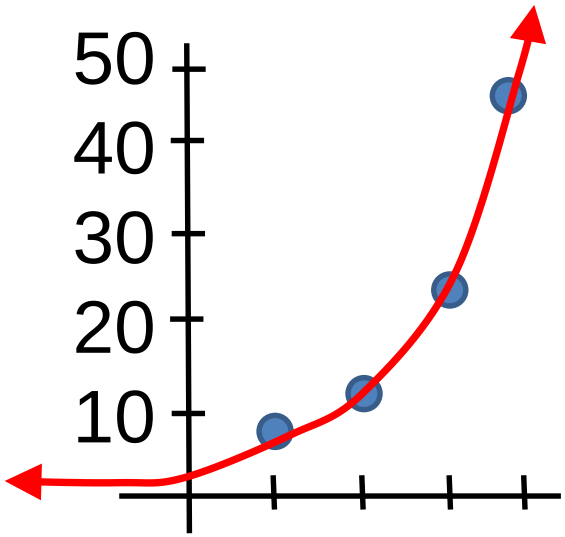
same as

"growth factor" of exponential function

How is geometric sequence different from the Exponential function?

$f(x) = 3(2)^x$

$a_n = 6(2)^{n-1}$ For: n = 1, 2, 3, 4



n	0	1	2	3	4
a_n	3	6	12	24	48

Different:

"domain" of geometric sequence is the natural numbers

"domain" of exponential function is all real numbers

Exponential Function

Geometric Sequence

$$f(x) = ab^x$$

$$a_n = a_1 * r^{n-1}$$

General Equation: Emphasizes the "0th term"

n	0	1	2	3	4
a_n	3	6	12	24	48

Explicit Formula: Emphasizes the 1st term

$$f(x) = 3(2)^x$$

$$a_n = 6(2)^{n-1}$$

Are they equivalent equations?

$$f(x) = 3(2)^x \qquad a_n = 6(2)^{n-1}$$

$$y = 3(2)^x \qquad y = 6(2)^{x-1}$$

By substitution (since y = y):

$$3(2)^x = 6(2)^{x-1}$$

Product of Powers Property

$$3(2)^x = 6(2)^{-1}(2)^x$$

$$x^2 x^3 = x^5$$

$$3(2)^x = 6\left(\frac{1}{2}\right)^1 (2)^x$$

Negative Exponent Property

$$3(2)^x = 3(2)^x$$

Yes, equivalent equations!!!

Your turn: Find both the recursive and explicit formulas for the following sequence of numbers.

n	1	2	3	4
a_n	5	15	45	135

Common ratio = 3

Recursive: $a_1 = 5 \quad a_n = 3a_{n-1}$ For: n = 2, 3, 4

Explicit: $\boxed{a_n = 5(3)^{n-1}}$ For: n ≥ 1

function $f(x) = \frac{5}{3}(3)^x$

Prove that these are the same.

Prove that these are the same.

$$f(x) = \frac{5}{2}(2)^x$$

$$a_n = 5(2)^{n-1}$$

$$y = 5 * \frac{1}{2} * (2)^x$$

$$y = 5(2)^{x-1}$$

$$y = 5 * \left(\frac{1}{2}\right)^1 * (2)^x$$

$$y = 5 * 2^{-1} * (2)^x$$

$$y = 5(2)^{x-1}$$

The sum of the first ‘n’ terms of a geometric sequence

$$Sum_n = S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

$$\sum_{k=1}^{n} a_1 * r^{k-1} = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$$

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$$

Multiply both sides by ‘r’

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^n$$

Subtract

$$S_n - rS_n = a_1 - a_1 r^n$$

$$(1 - r)S_n = (1 - r^n)a_1$$

$$\boxed{S_n = a_1 \left(\frac{1-r^n}{1-\mathrm{r}}\right)}$$

Find the sum of the first 'n' terms of the geometric sequence

$$a_n = a_1 * r^{n-1}$$

$$\{a_n\} = \left\{ (\frac{1}{4})^n \right\}$$

$$\boxed{S_n = a_1 \left(\frac{1-r^n}{1-\mathrm{r}}\right)}$$

$$S_n = \frac{1}{4}\left(\frac{1-\left(\frac{1}{4}\right)^n}{1-\left(\frac{1}{4}\right)}\right)$$

$$\{a_n\} = \left\{\frac{1}{4}(\frac{1}{4})^{n-1}\right\}$$

$$S_n = \frac{1}{4}\left(\frac{1-\left(\frac{1}{4}\right)^n}{\frac{3}{4}}\right)$$

$$S_n = \frac{1}{3}\left(1 - \left(\frac{1}{4}\right)^n\right)$$

$$S_n = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^n$$

Find the sum of the first 'n' terms of the geometric sequence

$$a_n = a_1 * r^{n-1}$$

$$\{a_n\} = \{3(2)^{n-1}\}$$

$$\boxed{S_n = a_1\left(\frac{1 - r^n}{1 - \mathrm{r}}\right)}$$

$$S_n = -3(1 - 2^n)$$

$$S_n = 3\left(\frac{1 - 2^{\,n}}{1 - 2}\right)$$

$$S_n = -3 + 3(2^n)$$

How big is the nth term as n →∞ $a_n = a_1 * r^{n-1}$

$a_n = 2^{n-1}$ $a_n \to \infty$ $a_n = \left(\frac{1}{2}\right)^{n-1}$ $a_n \to 0$

The sum of an <u>*infinite geometric sequence*</u> is called an <u>*Infinite Geometric SERIES*</u>:

$$\sum_{k=1}^{\infty} a_1 * r^{n-1}$$

The sum of the first 'n' terms of the geometric sequence

If $r^n \to 0$ then $S_n \to \frac{a_1}{1-r}$

$$\boxed{S_n = a_1\left(\frac{1-r^n}{1-\mathrm{r}}\right)}$$

$$\boxed{S_n = \frac{a_1}{1-\mathrm{r}} - \frac{a_1 r^n}{1-\mathrm{r}}}$$

$$\sum_{k=1}^{\infty} a_1 * r^{n-1}$$

$$\boxed{S_n = \frac{a_1}{1-\mathrm{r}} - \frac{a_1 r^n}{1-\mathrm{r}}}$$

An *Infinite Geometric SERIES* will approach a certain number if $|r| < 1$

$$\boxed{\lim_{n\to\infty} S_n = \frac{a_1}{1-\mathrm{r}}}$$

If $|r| < 1$ we say the *Infinite Geometric SERIES* ***converges***

to $\frac{a_1}{1-\mathrm{r}}$

$$\sum_{k=1}^{\infty} a_1 * r^{n-1}$$

Does the following series converge?

$$\sum_{k=1}^{\infty} 3\left(\frac{3}{4}\right)^{n-1} \qquad r = \frac{3}{4} \rightarrow \text{yes}$$

If $|r| < 1$ we say the *Infinite Geometric SERIES* ***converges***

to $\frac{a_1}{1-\mathrm{r}}$

What does the series converge to?

$$\sum_{k=1}^{\infty} 3\left(\frac{3}{4}\right)^{n-1} = \frac{3}{1-\frac{3}{4}}$$

$$\sum_{k=1}^{\infty} a_1 * r^{n-1} = \frac{a_1}{1-\mathrm{r}}$$

$$= \frac{3}{\frac{1}{4}} \qquad = 12$$

Repeating Decimals $0.9999\overline{9}$

$$= 0.9 + 0.09 + 0.009 + \cdots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$\sum_{k=1}^{\infty} a_1 * r^{n-1}$$

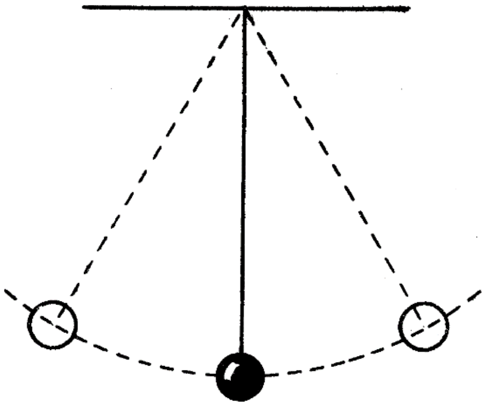
$$\sum_{k=1}^{\infty} \frac{9}{10^k} = \sum_{k=1}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^{n-1} = \frac{a_1}{1 - \mathrm{r}} = \frac{\frac{9}{10}}{1 - \frac{1}{10}}$$

$$= \frac{\frac{9}{10}}{\frac{9}{10}} = 1$$

Pendulums

A pendulum initially swings through an arc of 20 inches.

After each swing, the following arc is 0.95 times the arc length of the previous arc length.



Write the explicit formula for the arc length.

$$a_n = a_1 * r^{n-1} = 20(0.95)^{\mathrm{n}-1}$$

What is the 10th arc length?

$$a_{10} = 20(0.95)^{10-1} \qquad a_{10} = 12.6\ in$$

What is the total distance the pendulum traveled in 20 swings?

$$\sum_{k=1}^{\infty} 20 * (0.95)^{n-1}$$

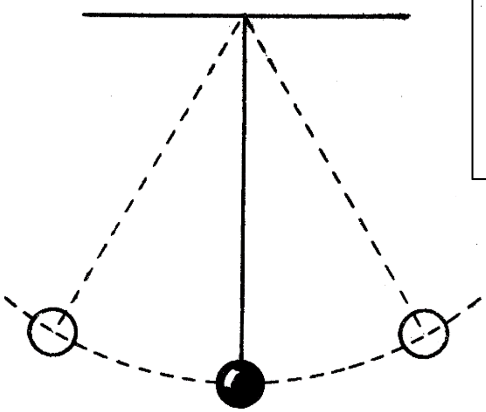
$$\boxed{S_n = a_1\left(\frac{1-r^n}{1-\mathrm{r}}\right)}$$

$$S_{20} = 20\left(\frac{1-(0.95)^{20}}{1-0.95}\right)$$

$$S_{20} = 256.6\ in$$

Pendulums

After how many swings will it have traveled 300 inches?



$$\boxed{S_n = a_1\left(\frac{1-r^n}{1-\mathrm{r}}\right)}$$

$$300 = 20\left(\frac{1-(0.95)^n}{1-0.95}\right)$$

$$300 = 20\left(\frac{1-(0.95)^n}{0.05}\right)$$

$$300 = 400(1-(0.95)^n)$$

$$300 = 400 - 400(0.95)^n)$$

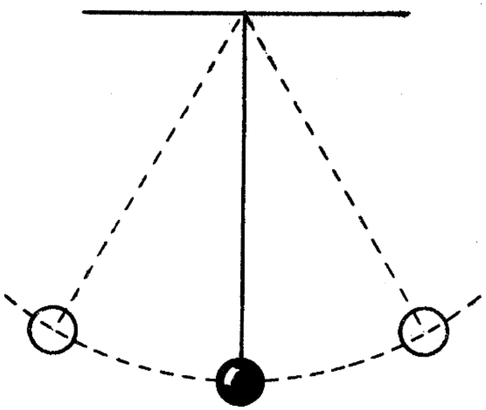
$$-100 = -400(0.95)^n$$

$$\frac{1}{4} = (0.95)^n$$

$$n = \log_{0.95}\frac{1}{4}$$

$$n = 27.03 \text{ swings}$$

Pendulums



When it stops swinging, what is the total distance that the pendulum has swung?

$$\sum_{k=1}^{\infty} a_1 * r^{n-1}$$

$$\sum_{k=1}^{\infty} 20 * (0.95)^{n-1} \quad = \frac{a_1}{1 - \mathrm{r}}$$

$$= \frac{20}{1 - 0.95} \quad = \frac{20}{0.05} \quad = 400\ \mathit{in}$$

<u>Annuity</u> Is a sequence of regularly paid deposits to an account that (hopefully) pays interest.

If the deposit occurs at the same time the interest is paid (daily, monthly, quarterly, etc.), the annuity is called an *<u>ordinary annuity</u>*.

For once per year deposits, the amount of money in an ordinary annuity is given by

$$A = P\,\frac{(1+i)^t - 1}{i}$$

here '*i*' is the annual interest rate, '*P*' is the deposit amount, and '*t*' *is the number of years.*

Similar to compound interest, for more frequently than one/year deposits, the amount would be given by:

$$A = P\,\frac{\left(1+\frac{i}{k}\right)^{kt} - 1}{\frac{i}{k}}$$

To avoid car payments, a wise investor decides to deposit money to an ordinary annuity that pays monthly interest at an annual rate of 3%.

How often will he/she deposit money to the annuity? monthly

How much should the investor deposit each month in order to have enough money to buy a $20,000 in 3 years?

$$A = P\frac{\left(1+\frac{i}{k}\right)^{kt} - 1}{\frac{i}{k}}$$

$$20000 = P\frac{\left(1+\frac{0.03}{12}\right)^{12(3)} - 1}{\frac{0.03}{12}}$$

$$20000 = P(37.62)$$

$$P = \$531.63$$