Math-1050
Session #32
(12.5: Partial Fraction Decomposition)

What you'll learn

- Partial Fraction Decomposition
- Denominators with Linear Factors
- Denominators with Irreducible Quadratic Factors

… and why

Partial fraction decompositions are used in calculus and are a powerful tool when integrating rational functions.

Adding rational expressions

$$\frac{1}{x+2}+\frac{2}{x-4}$$

Need a common denominator

$$\left(\frac{x-4}{x-4}\right)*\frac{1}{x+2}+\frac{2}{x-4}*\left(\frac{x+2}{x+2}\right)$$

$$\frac{x-4}{(x-4)(x+2)}+\frac{2(x+2)}{(x-4)(x+2)} \rightarrow \frac{3x}{x^2-2x-8}$$

Add these fractions $$\frac{1}{x+3}+\frac{3}{x-2}$$

$$\left(\frac{x-2}{x-2}\right)*\frac{1}{x+3}+\frac{3}{x-2}*\left(\frac{x+3}{x+3}\right)$$

$$\frac{x-2}{(x-2)(x+3)}+\frac{3(x+3)}{(x-2)(x+3)} \rightarrow \frac{4x+7}{x^2+x-6}$$

Going backwards.

$$\frac{3x}{x^2 - 2x - 8}$$

Can you break this apart into the two fractions that were added together?

$$\frac{3x}{(x-4)(x+2)}$$

Factor the denominator

$$\frac{3x}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

Write a general form sum of fractions.

$$\frac{3x}{(x-4)(x+2)} = \frac{A(x+2)}{(x-4)(x+2)} + \frac{B(x-4)}{(x-4)(x+2)}$$

Obtain a common denominator

$$3x = A(x+2) + B(x-4)$$

Eliminate the denominator

$$3x = A(x+2) + B(x-4)$$

Pick a “nice value” for x in order to solve for A. Let x = 4

$$3(4) = A(4+2) + B(4-4)$$ Solve for A

$$12 = A(6) \qquad A = 2$$

Pick another “nice value” for x in order to solve for B. Let x = -2

$$3(-2) = A(-2+2) + B(-2-4)$$ Solve for B

$$-6 = B(-6) \qquad B = 1$$

$$\frac{3x}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$\boxed{\frac{3x}{(x-4)(x+2)} = \frac{2}{x-4} + \frac{1}{x+2}}$$

We will cover 4 General Types: $\frac{f(x)}{d(x)}$

1. *d(x)* can be factored into *non-repeated* linear factors →
(ax + b)(cx + d) etc.,

2. If *d(x)* can be factored into *repeated* linear factors →
(ax + b)(ax + b) → $(ax+b)^2$

3. If *d(x)* is a *non-repeated* 2nd degree polynomial and cannot be factored → ax^2+bx+c

4. If *d(x)* is a *repeated* 2nd degree polynomial and cannot be factored → $(ax^2+bx+c)^2$

Partial Fraction Decomposition of $\dfrac{f(x)}{d(x)}$

If *d(x)* can be factored into *non-repeated linear factors* →

(ax + b)(cx + d) etc., This was our previous example

the partial fraction decomposition of $\dfrac{r(x)}{d(x)}$

will take the form: $\dfrac{r(x)}{d(x)} = \dfrac{A}{(ax+b)} + \dfrac{B}{(cx+d)} + \cdots$

Partial Fraction Decomposition of $\frac{f(x)}{d(x)}$

If *d(x)* can be factored into *repeated* linear factors →

(ax + b)(ax + b) → $(ax+b)^2$

the partial fraction decomposition of $\frac{r(x)}{d(x)}$

will take the form: $\frac{r(x)}{d(x)} = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \cdots$

$$\frac{9x+2}{x^2-2x+1}$$

Factor the denominator

$$\frac{9x+2}{(x-1)(x-1)}$$

$$\frac{9x+2}{(x-1)(x-1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Write a general form sum of fractions.

Obtain a common denominator

$$\frac{9x+2}{(x-1)(x-1)} = \left(\frac{x-1}{x-1}\right) * \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Eliminate the denominator

$$9x+2 = A(x-1) + B$$

$$9x + 2 = A(x - 1) + B$$

Pick a "nice value" for x in order to solve for B. Let x = 1

$$9(1) + 2 = A(1 - 1) + B$$

Solve for B

$$B = 11$$

Pick another "nice value" for x in order to solve for A. Let x = 0

$$9(0) + 2 = A(0 - 1) + (11)$$

Solve for A

$$2 = -A + 11$$

$$A = 9$$

$$\frac{9x + 2}{(x - 1)(x - 1)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2}$$

$$\boxed{\frac{9x + 2}{x^2 - 2x + 1} = \frac{9}{x - 1} + \frac{11}{(x - 1)^2}}$$

Partial Fraction Decomposition of $\dfrac{f(x)}{d(x)}$

If *d(x)* is a *non-repeated* 2nd degree polynomial and cannot be factored →

the partial fraction decomposition of $\dfrac{r(x)}{d(x)}$

will take the form: $$\frac{r(x)}{d(x)} = \frac{Ax + B}{(ax^2 + bx + c)}$$

$$\frac{2x+1}{x^3+8}$$

factor

$$\frac{2x+1}{(x+2)(x^2-2x+4)}$$

Write a general form sum of fractions.

$$\frac{2x+1}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{Bx+c}{x^2-2x+4}$$

Multiply by the common denominator

$$\frac{2x+1}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{(Bx+C)}{x^2-2x+4}$$

$$2x+1 = A(x^2-2x+4)+(Bx+C)(x+2)$$

Expand right side

$$2x+1 = Ax^2-2Ax+4A+Bx^2+2Bx+Cx+2C$$

$$2x + 1 = Ax^2 - 2Ax + 4A + Bx^2 + 2Bx + Cx + 2C$$

Rewrite the right side using common factors.

$$2x + 1 = (A + B)x^2 + (-2A + 2B + C)x + (4A + 2C)$$

Compare the left and right sides of the “=“ sign.

$$0x^2 + 2x + 1 = (A + B)x^2 + (-2A + 2B + C)x + 4A + 2C$$

$x^2\ term$: $A + B = 0$

$x^1\ term$: $-2A + 2B + C = 2$

$x^0\ term$: $4A + 2C = 1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 2 & 1 & 2 \\ 4 & 0 & 2 & 1 \end{array}\right)$$

Solve the system of equations

Cramer's Rule

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 2 & 1 & 2 \\ 4 & 0 & 2 & 1 \end{array}\right)$$

$$A = \frac{D_A}{D} \qquad B = \frac{D_B}{D} \qquad C = \frac{D_C}{D}$$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$D = 1\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} + 0\begin{vmatrix} -2 & 2 \\ 4 & 0 \end{vmatrix} \qquad D = 12$$

$$D_A = 0\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} \qquad D_A = -3$$

$$A = \frac{-3}{12} \qquad \boxed{A = \frac{-1}{4}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 2 & 1 & 2 \\ 4 & 0 & 2 & 1 \end{array}\right)$$

$$A = \frac{D_A}{D} \qquad B = \frac{D_B}{D} \qquad C = \frac{D_C}{D}$$

$$\boxed{A = \frac{-1}{4}} \qquad D = 12$$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$D_B = 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} + 0\begin{vmatrix} -2 & 2 \\ 4 & 1 \end{vmatrix}$$

$$D_B = 5 \qquad \boxed{B = \frac{5}{12}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 2 & 1 & 2 \\ 4 & 0 & 2 & 1 \end{array}\right)$$

$$A = \frac{D_A}{D} \qquad B = \frac{D_B}{D} \qquad C = \frac{D_C}{D}$$

$$\boxed{A = \frac{-1}{4}} \qquad \boxed{B = \frac{5}{12}} \qquad D = 12$$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$D_C = 1\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} -1\begin{vmatrix} -2 & 2 \\ 4 & 1 \end{vmatrix} +0\begin{vmatrix} -2 & 2 \\ 4 & 0 \end{vmatrix}$$

$$D_C = 2 + 10 \qquad D_C = 12 \qquad C = \frac{12}{12} \qquad \boxed{C = 1}$$

Putting it all together.

$$\frac{2x+1}{x^3+8}$$

$$\frac{2x+1}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{Bx+c}{x^2-2x+4}$$

$$\boxed{A = \frac{-1}{4}} \quad \boxed{B = \frac{5}{12}} \quad \boxed{C = 1}$$

$$\frac{2x+1}{(x+2)(x^2-2x+4)} = \frac{-\frac{1}{4}}{x+2} + \frac{\frac{5}{12}x+c}{x^2-2x+4}$$

Partial Fraction Decomposition of $\dfrac{f(x)}{d(x)}$

If *d(x)* is a ***repeated*** 2^{nd} degree polynomial and cannot be factored →

the partial fraction decomposition of $\dfrac{r(x)}{d(x)}$

will take the form:

$$\frac{r(x)}{d(x)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{Cx+D}{(ax^2+bx+c)^2} + \cdots$$

$$\frac{x^2+1}{(x^2+3)^2}$$

Write a general form sum of fractions.

$$\frac{x^2+1}{(x^2+3)^2} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{(x^2+3)^2}$$

Multiply by the common denominator

$$x^2+1 = (Ax+B)(x^2+3) + Cx + D$$

Expand right side

$$x^2+1 = Ax^3 + Bx^2 + 3Ax + 3B + Cx + D$$

Rewrite the right side using common factors.

$$x^2+1 = Ax^3 + Bx^2 + (3A+C)x + (3B+D)$$

$$x^2 + 1 = Ax^3 + Bx^2 + (3A + C)x + (3B + D)$$

Compare the left and right sides of the "=" sign.

x^3 $term$: $A = 0$

x^2 $term$: $B = 1$

x^1 $term$: $4A + C = 0$

x^0 $term$: $3B + D = 1$

Solve the system of equations

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 4 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 4 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 \end{array}\right)$$

Solve the system of equations

$$R_{new3} = -4 * R_{old4} + R_{old3}$$

$$R_{new4} = -3 * R_{old2} + R_{old4}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

$$\boxed{D = -2}$$

$$\boxed{C = 0}$$

$$\boxed{B = 1}$$

$$\boxed{A = 0}$$

$$\frac{x^2+1}{(x^2+3)^2}$$

Putting it all together

$$D = -2$$

$$C = 0$$

$$B = 1$$

$$A = 0$$

$$\frac{x^2+1}{(x^2+3)^2} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{(x^2+3)^2}$$

$$\frac{x^2+1}{(x^2+3)^2} = \frac{1}{x^2+3} + \frac{-2}{(x^2+3)^2}$$

1. *Denominator* can be factored into *non-repeated* linear factors

$$\frac{f(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

$$\frac{f(x)}{x(ax+b)} = \frac{A}{x} + \frac{B}{ax+b}$$

2. *Denominator* is non-repeated linear factor and non-repeated non-factorable quadratic.

$$\frac{f(x)}{x(ax^2+bx+c)} = \frac{A}{x} + \frac{Bx+C}{ax^2+bx+c}$$

3. Denominator can be factored into a combination of *repeated* linear factors and *non-repeated* linear factors.

$$\frac{f(x)}{x^2(ax+b)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{ax+b}$$

4. Denominator can be factored into *repeated* linear factors

$$\frac{f(x)}{x^2(ax+b)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{ax+b} + \frac{D}{(ax+b)^2}$$

Decompose the fraction: $\frac{5x-1}{x^2-2x-15}$

$$\frac{5x-1}{(x+3)(x-5)}$$

Factor the denominator

$$\frac{5x-1}{(x+3)(x-5)} = \frac{A}{x+3} + \frac{B}{x-5}$$

Write a general form sum of fractions.

$$5x-1 = A(x-5) + B(x+3)$$

Multiply by the common Denominator.

$$5x-1=A(x-5)+B(x+3)$$

Pick a "nice value" for x in order to solve for A. Let x = -3

$$5(-3)-1=A(-3-5)+B(-3+3)$$

Solve for A

$$-16=A(-8) \qquad A=2$$

Pick another "nice value" for x in order to solve for B. Let x = 5

$$5(5)-1=A(-5+5)+B(5+3)$$

Solve for B

$$24=B(8) \qquad B=3$$

$$\frac{5x-1}{(x+3)(x-5)}=\frac{A}{x+3}+\frac{B}{x-5} \qquad =\frac{2}{x+3}+\frac{3}{x-5}$$

Decompose the following fraction $\dfrac{x+22}{x^2+2x-8}$

$$\frac{x+22}{(x+4)(x-2)}$$

Factor the denominator

$$\frac{x+22}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}$$

Write a general form sum of fractions.

$$x+22 = A(x-2) + B(x+4)$$

Eliminate the denominator

Pick a "nice value" for x in order to solve for A.

Let x = -4

$$-4+22 = A(-4-2) + B(-4+4)$$

$$A = -3$$

Pick another "nice value" for x in order to solve for B.

Let x = 2

$$x+22 = A(x-2) + B(x+4)$$

$$2+22 = A(2-2) + B(2+4)$$

Solve for B $B = 4$

$$\frac{x+22}{(x+4)(x-2)} = \frac{-3}{(x+4)} + \frac{4}{(x-2)}$$

$$\frac{2x+5}{(x+1)^2} \qquad \frac{A}{(x+1)^1} + \frac{B}{(x+1)^2}$$

$$\frac{2x+5}{(x+1)^2} = \left(\frac{x+1}{x+1}\right) * \frac{A}{(x+1)} + \frac{B}{(x+1)^2}$$

Eliminate the denominator

$$2x+5 = A(x+1) + B$$

Pick and “nice number” for x to make it easy to solve for B.

$$2(-1)+5 = A(-1+1) + B$$

Let x = -1

$$3 = B$$

$$2x+5 = Ax + A + 3$$

Compare left/right

$$2x = A \qquad 5 = A+3 \qquad A = 2$$

$$\frac{2x+5}{(x+1)^2} = \frac{2}{(x+1)^1} + \frac{3}{(x+1)^2}$$

$$\frac{2x^2+3x+1}{x^3+x}$$

$$\frac{2x^2+3x+1}{x(x^2+1)} = \frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$2x^2+3x+1 = A(x^2+1)+Bx^2+Cx$$

Eliminate the denominator

$$2x^2+3x+1 = Ax^2+Bx^2+Cx+A$$

$$2x^2+3x+1 = (A+B)x^2+Cx+A$$

x^2 *term*: $A+B=2$

Solve the system of equations

B = 1

x^1 *term*: $C=3$

x^0 *term*: $A=1$

$$\frac{2x^2+3x+1}{x^3+x} = \frac{1}{x}+\frac{x+3}{x^2+1}$$