Math-1050

Session #31
12.4 (Matrix Algebra)

Matrix: A rectangular arrangement of numbers in rows and columns.

Dimension (order): Of a matrix with 3 rows and 2 columns is: 3 x 2

In general we say: m x n where:

"m" = # of rows

"n" = # of Columns

$2x - 3y \longrightarrow$

$7x + 2y \longrightarrow$

Matrix of coefficients

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix}$$

$2x - 3y = 8 \longrightarrow$

$7x + 2y = 2 \longrightarrow$

Augmented Matrix

$$\left(\begin{array}{cc|c} 2 & -3 & 8 \\ 7 & 2 & 2 \end{array}\right)$$

Each **element**, or **entry**, a_{ij}, of the matrix uses <u>double subscript</u> notation.

<u>Row subscript</u>: is the 1st letter (*i*)
<u>Column Subscript</u>: is the 2nd letter (j)

<u>Example</u>: Element *aij* is in the *i*th row and *j*th column.

$$[a_{ij}]$$

Is "shorthand" for a matrix with 'i' rows and 'j' columns. ('i' x 'j')

Elements: numbers in the matrix

What is the "order" of this matrix ?

$$\begin{pmatrix} \underset{1,1}{\underline{5}} & \underset{1,2}{\underline{-2}} \\ \underset{2,1}{\underline{3}} & \underset{2,2}{\underline{1}} \end{pmatrix}$$

Equal matrices: have same "order" and each corresponding element is equal.

1. What number is $a_{2,1}$ 3
2. What number is $a_{1,2}$ -2

Matrices can be <u>HUGE</u> !

5 columns

4 rows

$$\begin{pmatrix} 3 & 4 & 5 & 7 & 8 \\ 9 & 1 & 3 & 6 & 2 \\ 7 & -2 & 5 & -1 & 7 \\ -3 & 3 & -5 & 2 & 8 \end{pmatrix}$$

Dimension: m rows x n columns

4 x 5

Scalar: A real number (a constant) that is multiplied by every element in the matrix.

Scalar Multiplication: The process of multiplying every element in the matrix by a scalar (constant).

Let: 'A' represent the matrix $[a_{ij}]$

Then: 3A = $3[a_{ij}]$

Multiplying by a constant (also called a "Scalar Multiplication")

$$5\begin{pmatrix} 3 & 4 \\ 9 & 1 \\ 7 & -2 \end{pmatrix} = \begin{pmatrix} 5(3) & 5(4) \\ 5(9) & 5(1) \\ 5(7) & 5(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 15 & 20 \\ 45 & 5 \\ 35 & -10 \end{pmatrix}$$

Basic Operations: Addition

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ -3 & 5 \end{pmatrix} = ? \begin{pmatrix} \underline{A_{11}+B_{11}} & \underline{A_{12}+B_{12}} \\ \underline{A_{21}+B_{21}} & \underline{A_{22}+B_{22}} \end{pmatrix}$$

A + B

$$\begin{pmatrix} \underline{2+3} & \underline{-3+1} \\ \underline{7-3} & \underline{2+5} \end{pmatrix}$$

$$\begin{pmatrix} \underline{5} & \underline{-2} \\ \underline{4} & \underline{7} \end{pmatrix}$$

Matrices are added "corresponding element" to "corresponding element"

Add the matrices

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} \underline{5} & \underline{-2} \\ \underline{4} & \underline{7} \end{pmatrix}$$

Basic Operations: Addition

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 7 \\ -3 & 5 & 6 \end{pmatrix} = ?$$

CAN'T DO THIS!!!!!!

(must be the same order for addition/subtraction)

Matrix Multiplication

Finding the *Order* of the "Product" of Two Matrices
(2 matrices multiplied by each other):

Matrix A x Matrix B = AB

m x n (times) n x p = m x p

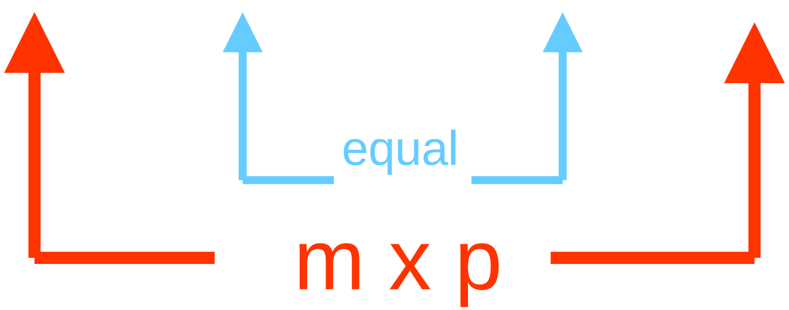
Must be equal !!

m x p

What is the dimension of the product?

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 5 \\ -3 & 5 & 4 \end{pmatrix} = ?$$

$$2 \times 2 \qquad 2 \times 3 \qquad = 2 \times 3$$

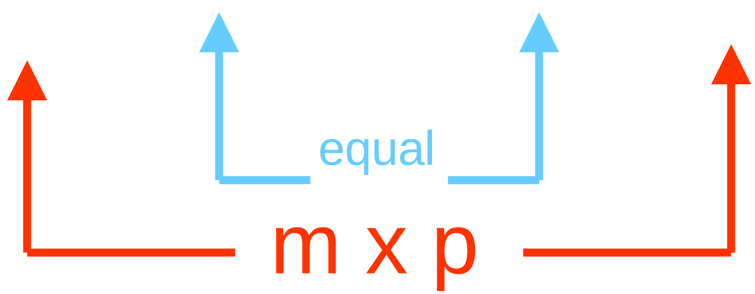


What is the dimension of the product?

$$A \quad \times \quad B \quad = \quad ?$$

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 1 \\ -3 & 5 \end{pmatrix} = ?$$

$$2 \times 2 \qquad 2 \times 2 \quad = \quad 2 \times 2$$



What is the dimension of the product?

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 1 \\ -3 & 5 \\ 4 & 6 \end{pmatrix} = ?$$

2 x 2 3 x 2

NOT equal !!!!!!!

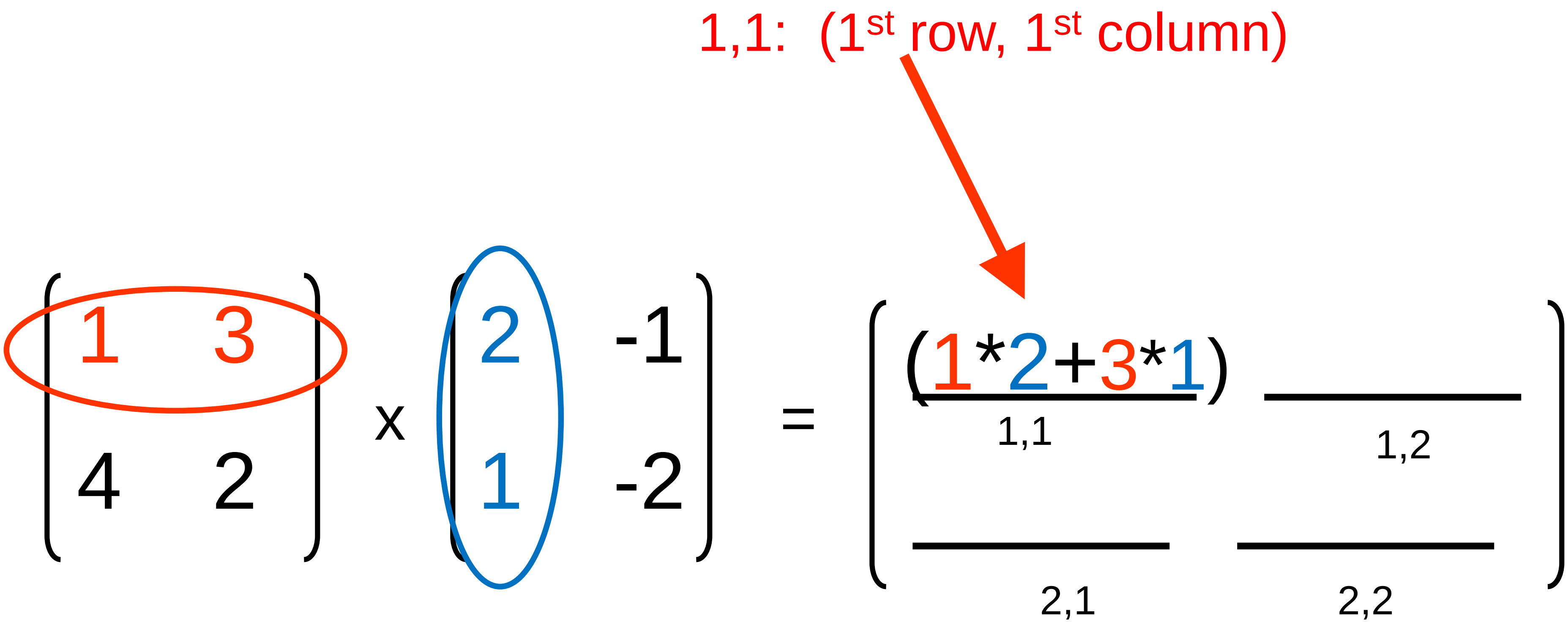
<u>CAN'T DO THIS!!!!!!</u>

So, how do you multiply matrices?

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \underset{1,1}{\_\_\_\_\_\_} & \underset{1,2}{\_\_\_\_\_\_} \\ \underset{2,1}{\_\_\_\_\_\_} & \underset{2,2}{\_\_\_\_\_\_} \end{pmatrix}$$

1. Can you multiply?

2. What is order of the answer matrix?



The “address” of the answer explains what elements are multiplied.

So, how do you multiply matrices?

$$\begin{pmatrix}1 & 3\\4 & 2\end{pmatrix} \times \begin{pmatrix}2 & -1\\1 & -2\end{pmatrix} = \begin{pmatrix} \underset{1,1}{\underline{\quad 5 \quad}} & \underset{1,2}{\underline{(1*-1+3*-2)}} \\ \underset{2,1}{\underline{\qquad\quad}} & \underset{2,2}{\underline{\qquad\quad}}\end{pmatrix}$$

So, how do you multiply matrices?

$$\begin{pmatrix}1 & 3\\ 4 & 2\end{pmatrix} \times \begin{pmatrix}2 & -1\\ 1 & -2\end{pmatrix} = \begin{pmatrix}\underset{1,1}{\underline{\;5\;}} & \underset{1,2}{\underline{\;-7\;}}\\ \underset{2,1}{\underline{(4*2+2*1)}} & \underset{2,2}{\underline{\qquad}}\end{pmatrix}$$

So, how do you multiply matrices?

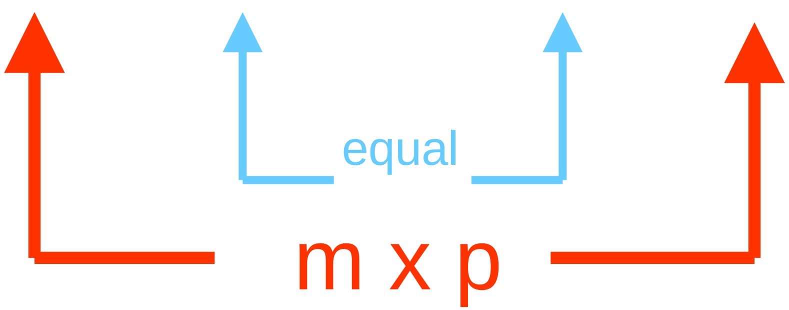
$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \underset{1,1}{\underline{5}} & \underset{1,2}{\underline{-7}} \\ \underset{2,1}{\underline{10}} & \underset{2,2}{\underline{(4*-1+2*-2)}} \end{pmatrix}$$

Your turn:

5. Write the product of the two matrices.

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 5 \\ -3 & 5 & 4 \end{pmatrix} = ?$$

2 x 2   2 x 3   = 2 x 3



Is Matrix Multiplication commutative?

A * B = AB

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -7 \\ 10 & -8 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} \times \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = ?$$

B * A = AB

Your Turn:

7. Write the product of the two matrices.

8. Does AB = BA ?

Matrix Equation

Matrix of variables

$$2x - 3y = 8$$
$$7x + 2y = 2$$

$$\begin{bmatrix} 2 & -3 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

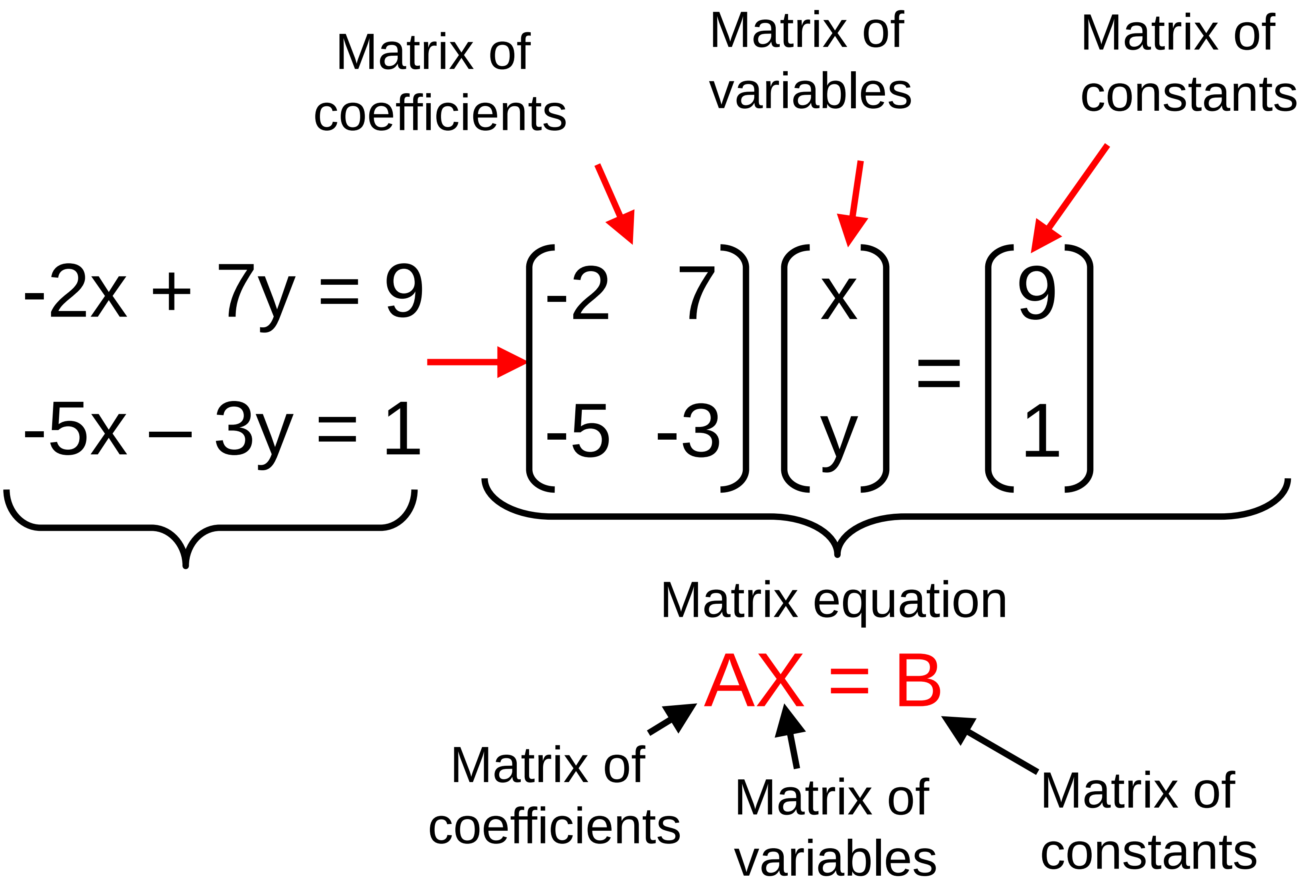
System of equations

Matrix of coefficients

Matrix of constants

Matrix equation

Generalized version of a matrix equation:



Inverse Property of Multiplication

Any number multiplied by its inverse (reciprocal) equals ‘1’.

$$3 * \frac{1}{3} = 1 \qquad 3 * 3^{-1} = 1$$

Any number multiplied by its inverse equals ‘1’ .

Square matrices can have inverses too.

What happens if you multiply a matrix by its inverse?

You get the “identity” matrix (1’s down the diagonal and zeroes everywhere else.

Inverse Matrices

Any square matrix multiplied by its inverse equals the identity matrix.

$$AA^{-1} = I_n$$

Solving a System of equation using Inverse Matrices:

Matrix equation

$$AX = B$$

Multiply (left/right) by the inverse matrix A.

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

The Identity Matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Is a “square” matrix, with 1’s down the “main diagonal and 0’s everywhere else.

Finding An Inverse Matrix

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 13 \end{pmatrix} \rightarrow \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 4 & 13 & 0 & 1 \end{array}\right)$$

1) Write the matrix and, to its right, write the identity matrix.

2) We need a 1 in the upper left position. It's already there.

3) Perform row operations on the left side matrix until you get zeroes below the main diagonal.

$$R_2 = R_2 - 4R_1 \quad \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{array}\right)$$

4) Perform row operations on the left side matrix until you get zeroes above the main diagonal.

$$R_1 = R_1 - 3R_2 \quad \left(\begin{array}{cc|cc} 1 & 0 & 13 & -3 \\ 0 & 1 & -4 & 1 \end{array}\right)$$

$$A^{-1} = \begin{bmatrix} 13 & -3 \\ -4 & 1 \end{bmatrix}$$

Finding An Inverse Matrix

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 13 \end{pmatrix} \rightarrow \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 4 & 13 & 0 & 1 \end{array}\right)$$

1) Write the matrix and, to its right, write the identity matrix.

2) Perform row operations on the left side matrix until you get zeroes below the main diagonal.

$$R_2 = R_2 - 4R_1 \qquad \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{array}\right)$$

3) Perform row operations on the left side matrix until you get zeroes above the main diagonal.

$$R_1 = R_1 - 3R_2 \qquad \left(\begin{array}{cc|cc} 1 & 0 & 13 & -3 \\ 0 & 1 & -4 & 1 \end{array}\right) \qquad A^{-1} = \begin{bmatrix} 13 & -3 \\ -4 & 1 \end{bmatrix}$$

Using Inverse Matrices to Solve Linear Systems

$$2x - 7y = -21$$
$$-x + 4y = 12$$

Step 1: Convert to a matrix equation

A X = B

$$\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

$$A \quad X = B$$

$$\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

$$A^{-1} A \quad X = A^{-1} B$$

$$X = A^{-1} B$$

<u>Step 2</u>: We need to find A^{-1}

$$A \quad X = B$$

$$\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

$$A^{-1} A \quad X = A^{-1} B$$

$$X = A^{-1} B$$

<u>Step 2</u>: We need to find A^{-1}

Step 3: Find the Inverse Matrix

$$A = \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \rightarrow \left(\begin{array}{cc|cc} 2 & -7 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{array}\right)$$

1) Write the matrix and, to its right, write the identity matrix.

2) We need a '1' in the upper left position.

$R_1 \rightarrow R_2$

$R_2 \rightarrow R_1$

$$\left(\begin{array}{cc|cc} -1 & 4 & 0 & 1 \\ 2 & -7 & 1 & 0 \end{array}\right) \quad R_1 \rightarrow -R_1 \quad \left(\begin{array}{cc|cc} 1 & -4 & 0 & -1 \\ 2 & -7 & 1 & 0 \end{array}\right)$$

3) Perform row operations on the left side matrix until you get zeroes below and then above the main diagonal.

$R_2 = R_2 - 2R_1$

$$\left(\begin{array}{cc|cc} 1 & -4 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$R_1 = R_1 + 4R_2$

$$\left(\begin{array}{cc|cc} 1 & 0 & 4 & 7 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$$A^{-1} = \begin{bmatrix} 13 & -3 \\ -4 & 1 \end{bmatrix}$$

Step 4: matrix multiplication

$$A^{-1} A \quad X = A^{-1} B$$

$$\begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} (4)(2)+(7)(-1) & (4)(-7)+((7)(4) \\ (1)(2)+(2)(-1) & (1)(2)+(2)(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1} A \quad X = A^{-1} B$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (4)(-21)+(7)(12) \\ (1)(-21)+(2)(12) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$1 \quad X = A^{-1} \quad B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

Your turn: multiply the right side.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$