

Math-1050

Session #31

12.4 (Matrix Algebra)

Matrix: A rectangular arrangement of numbers in rows and columns.

Dimension (order): Of a matrix with 3 rows and 2 columns is: 3 x 2

In general we say: m x n where:

“m” = # of rows

“n” = # of Columns

$$2x - 3y$$

$$7x + 2y$$



Matrix of
coefficients

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix}$$

Augmented Matrix

$$2x - 3y = 8$$

$$7x + 2y = 2$$



$$\begin{pmatrix} 2 & -3 & | & 8 \\ 7 & 2 & | & 2 \end{pmatrix}$$

Each **element**, or **entry**, a_{ij} , of the matrix uses double subscript notation.

Row subscript: is the 1st letter (i)

Column Subscript: is the 2nd letter (j)

Example: Element a_{ij} is in the i th row and j th column.

$[a_{ij}]$ Is “shorthand” for a matrix with ‘ i ’ rows and ‘ j ’ columns. (i x j)

Elements: numbers in the matrix

What is the
“order” of this
matrix ?

$$\begin{pmatrix} \frac{5}{1,1} & \frac{-2}{1,2} \\ \frac{3}{2,1} & \frac{1}{2,2} \end{pmatrix}$$

Equal matrices: have same “order” and each corresponding element is equal.

1. What number is $a_{2,1}$ **3**

2. What number is $a_{1,2}$ **-2**

Matrices can be HUGE !

5 columns

4 rows

$$\begin{pmatrix} 3 & 4 & 5 & 7 & 8 \\ 9 & 1 & 3 & 6 & 2 \\ 7 & -2 & 5 & -1 & 7 \\ -3 & 3 & -5 & 2 & 8 \end{pmatrix}$$

Dimension: m rows x n columns

4 x 5

Scalar: A real number (a constant) that is multiplied by every element in the matrix.

Scalar Multiplication: The process of multiplying every element in the matrix by a scalar (constant).

Let: 'A' represent the matrix $[a_{ij}]$

Then: $3A = 3[a_{ij}]$

Multiplying by a constant (also called a “Scalar Multiplication”)

$$5 \begin{pmatrix} 3 & 4 \\ 9 & 1 \\ 7 & -2 \end{pmatrix} = \begin{pmatrix} 5(3) & 5(4) \\ 5(9) & 5(1) \\ 5(7) & 5(-2) \end{pmatrix}$$
$$= \begin{pmatrix} 15 & 20 \\ 45 & 5 \\ 35 & -10 \end{pmatrix}$$

Basic Operations: Addition

$$\begin{pmatrix} \textcircled{2} & -3 \\ 7 & 2 \end{pmatrix} + \begin{pmatrix} \textcircled{3} & 1 \\ -3 & 5 \end{pmatrix} = ? \begin{pmatrix} \underline{\textcircled{A_{11} + B_{11}}} & \underline{A_{12} + B_{12}} \\ \underline{A_{21} + B_{21}} & \underline{A_{22} + B_{22}} \end{pmatrix}$$

A + B

Matrices are added
“corresponding element” to
“corresponding element”

$$\begin{pmatrix} \underline{\textcircled{2+3}} & \underline{-3+1} \\ \underline{7-3} & \underline{2+5} \end{pmatrix} = \begin{pmatrix} \underline{\textcircled{5}} & \underline{-2} \\ \underline{4} & \underline{7} \end{pmatrix}$$

Add the matrices

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} \underline{5} & \underline{-2} \\ \underline{4} & \underline{7} \end{pmatrix}$$

Basic Operations: Addition

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 7 \\ -3 & 5 & 6 \end{pmatrix} = ?$$

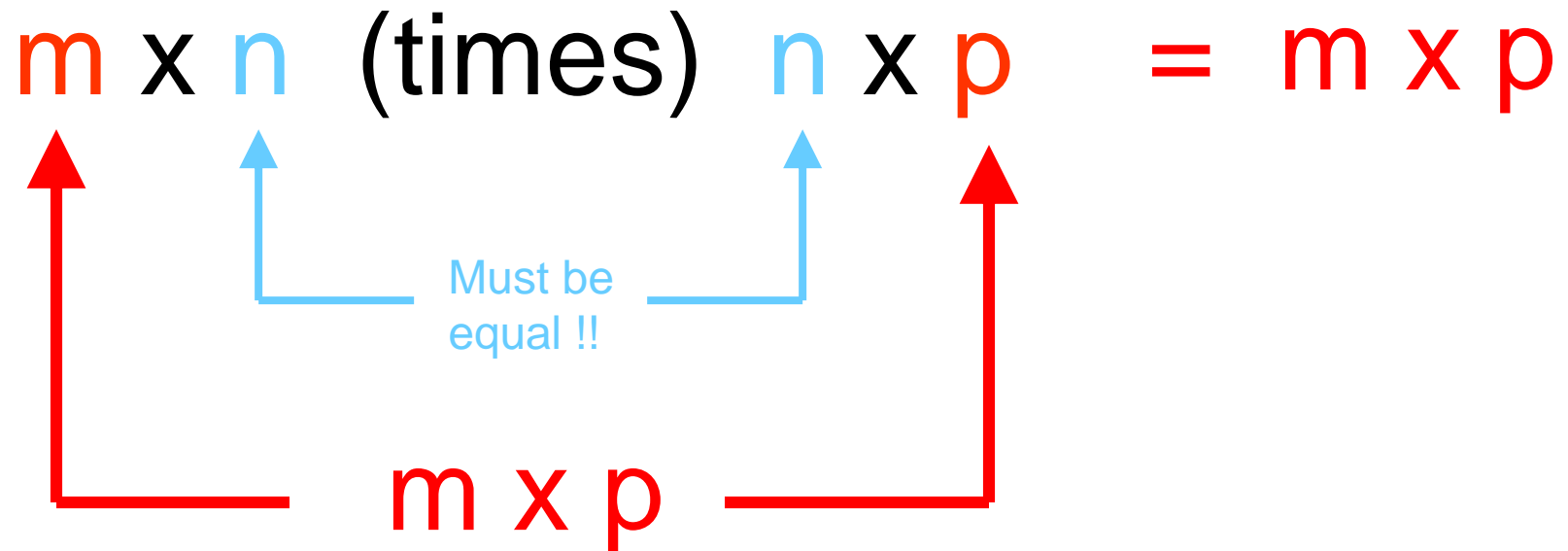
CAN'T DO THIS!!!!!!

(must be the same order for addition/subtraction)

Matrix Multiplication

Finding the Order of the “Product” of Two Matrices
(2 matrices multiplied by each other):

$$\text{Matrix A} \times \text{Matrix B} = AB$$



What is the dimension of the product?

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 5 \\ -3 & 5 & 4 \end{pmatrix} = ?$$

$2 \times 2 \quad 2 \times 3 = 2 \times 3$

equal

$m \times p$

What is the dimension of the product?

$$A \quad \times \quad B \quad = \quad ?$$
$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 1 \\ -3 & 5 \end{pmatrix} = ?$$

$$2 \times 2 \quad 2 \times 2 \quad = \quad 2 \times 2$$

Diagram illustrating the dimensions of the matrices and the resulting product:

- Matrix A is 2×2 (rows m , columns p).
- Matrix B is 2×2 (rows p , columns n).
- The resulting product matrix is 2×2 .

What is the dimension of the product?

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 1 \\ -3 & 5 \\ 4 & 6 \end{pmatrix} = ?$$

2 x 2 3 x 2

↑ ↑
NOT
equal
!!!!!!

CAN'T DO THIS!!!!!!

So, how do you multiply matrices?

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \overline{\hspace{2cm}} & \overline{\hspace{2cm}} \\ \overline{\hspace{2cm}} & \overline{\hspace{2cm}} \end{pmatrix}$$

$\begin{matrix} 1,1 & 1,2 \\ 2,1 & 2,2 \end{matrix}$

1. Can you multiply?
2. What is order of the answer matrix?

1,1: (1st row, 1st column)

The diagram illustrates the calculation of the element at row 1, column 1 of the product matrix. The first matrix has its first row (1, 3) circled in red. The second matrix has its first column (2, 1) circled in blue. An arrow points from the text '1,1: (1st row, 1st column)' to the first element of the result matrix. The result matrix shows the calculation (1*2 + 3*1) for the element at row 1, column 1, with horizontal lines below it. The other elements of the result matrix are also indicated by horizontal lines.

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{(1*2 + 3*1)}{1,1} & \frac{\quad}{1,2} \\ \frac{\quad}{2,1} & \frac{\quad}{2,2} \end{pmatrix}$$

The “address” of the answer explains what elements are multiplied.

So, how do you multiply matrices?

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{5}{1,1} & (1 * -1 + 3 * -2) \\ \frac{}{2,1} & \frac{}{2,2} \end{pmatrix}$$

The image illustrates the multiplication of two matrices. The first matrix is $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, with the top row $(1, 3)$ circled in red. The second matrix is $\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}$, with the second column $(-1, -2)$ circled in blue. The result is a 2×2 matrix. The top-left element is $\frac{5}{1,1}$. The top-right element is $(1 * -1 + 3 * -2)$, with the 1 in red, -1 in blue, and 3 in red. The bottom-right element is $\frac{}{2,2}$, with the $1,2$ label circled in red. The bottom-left element is $\frac{}{2,1}$.

So, how do you multiply matrices?

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{5}{1,1} & \frac{-7}{1,2} \\ \frac{(4*2+2*1)}{2,1} & \frac{}{2,2} \end{pmatrix}$$

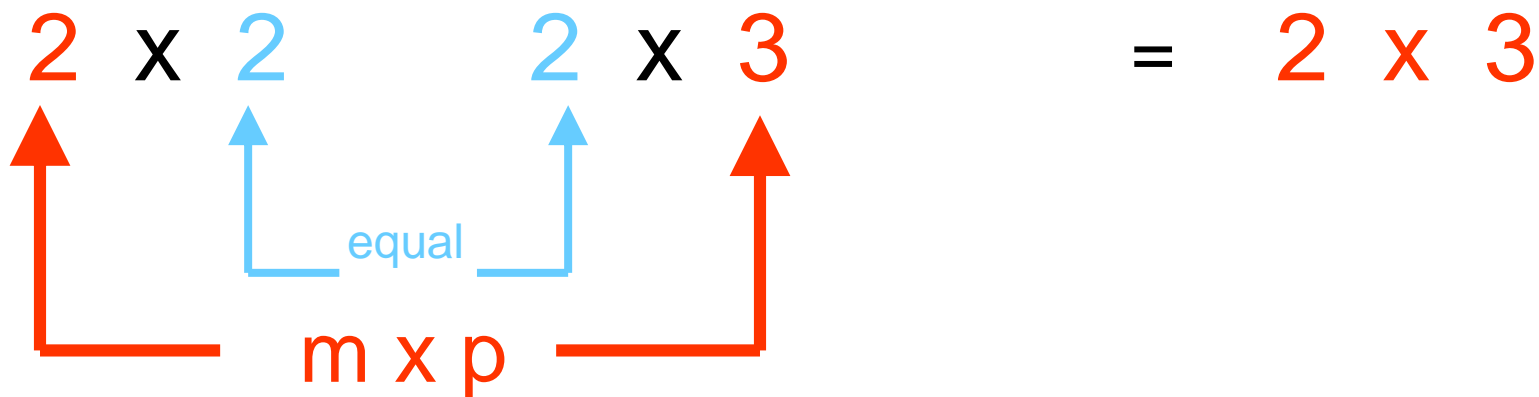
So, how do you multiply matrices?

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{5}{1,1} & \frac{-7}{1,2} \\ \frac{10}{2,1} & \frac{(4^* - 1 + 2^* - 2)}{2,2} \end{pmatrix}$$

Your turn:

5. Write the product of the two matrices.

$$\begin{pmatrix} 2 & -3 \\ 7 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 5 \\ -3 & 5 & 4 \end{pmatrix} = ?$$



Is Matrix Multiplication commutative?

$$A * B = AB$$

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -7 \\ 10 & -8 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} \times \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = ?$$

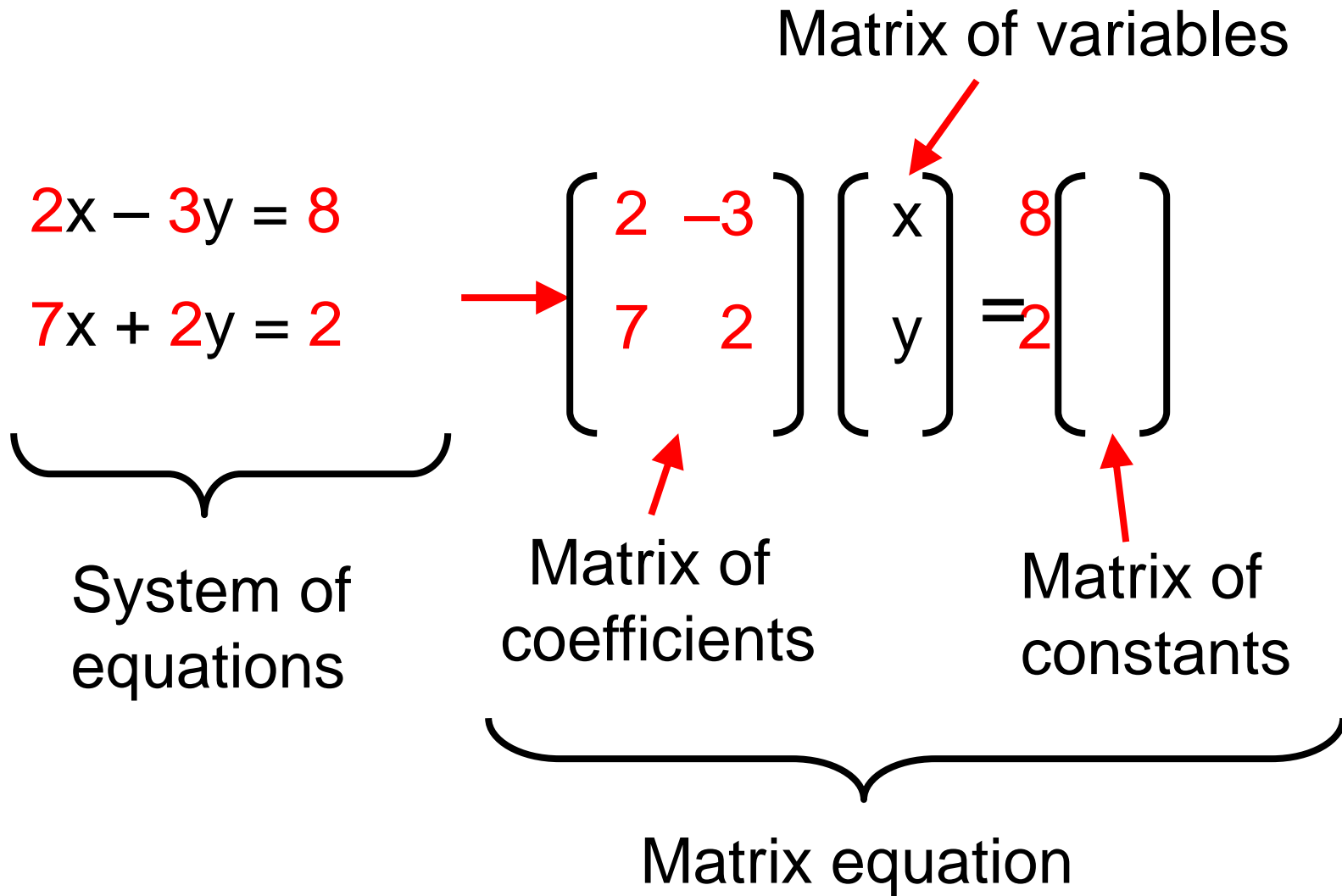
$$B * A = AB$$

Your Turn:

7. Write the product of the two matrices.

8. Does $AB = BA$?

Matrix Equation



Generalized version of a matrix equation:

Matrix of
coefficients

Matrix of
variables

Matrix of
constants

$$\begin{cases} -2x + 7y = 9 \\ -5x - 3y = 1 \end{cases} \rightarrow \begin{bmatrix} -2 & 7 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$$

Matrix equation

$$\mathbf{A} \mathbf{X} = \mathbf{B}$$

Matrix of coefficients Matrix of variables Matrix of constants

Inverse Property of Multiplication

Any number multiplied by its inverse (reciprocal) equals '1'.

$$3 * \frac{1}{3} = 1 \quad 3 * 3^{-1} = 1$$

Any number multiplied by its inverse equals '1' .

Square matrices can have inverses too.

What happens if you multiply a matrix by its inverse?

You get the “identity” matrix (1's down the diagonal and zeroes everywhere else).

Inverse Matrices

Any square matrix multiplied by its inverse equals the identity matrix.

$$AA^{-1} = I_n$$

Solving a System of equation using Inverse Matrices:

Matrix equation

$$AX = B$$

Multiply (left/right) by the inverse matrix A.

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

The Identity Matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Is a “square” matrix, with 1’s down the “main diagonal and 0’s everywhere else.

Finding An Inverse Matrix

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 13 \end{pmatrix} \rightarrow \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 4 & 13 & 0 & 1 \end{array} \right) \quad \begin{array}{l} 1) \text{ Write the matrix and, to its} \\ \text{right, write the identity matrix.} \end{array}$$

2) We need a 1 in the upper left position. It's already there.

3) Perform row operations on the left side matrix until you get zeroes below the main diagonal.

$$R_2 = R_2 - 4R_1 \quad \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{array} \right)$$

4) Perform row operations on the left side matrix until you get zeroes above the main diagonal.

$$R_1 = R_1 - 3R_2 \quad \left(\begin{array}{cc|cc} 1 & 0 & 13 & -3 \\ 0 & 1 & -4 & 1 \end{array} \right) \quad A^{-1} = \begin{bmatrix} 13 & -3 \\ -4 & 1 \end{bmatrix}$$

Finding An Inverse Matrix

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 13 \end{pmatrix} \rightarrow \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 4 & 13 & 0 & 1 \end{array} \right)$$

1) Write the matrix and, to its right, write the identity matrix.

2) Perform row operations on the left side matrix until you get zeroes below the main diagonal.

$$R_2 = R_2 - 4R_1 \quad \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{array} \right)$$

3) Perform row operations on the left side matrix until you get zeroes above the main diagonal.

$$R_1 = R_1 - 3R_2 \quad \left(\begin{array}{cc|cc} 1 & 0 & 13 & -3 \\ 0 & 1 & -4 & 1 \end{array} \right) \quad A^{-1} = \begin{bmatrix} 13 & -3 \\ -4 & 1 \end{bmatrix}$$

Using Inverse Matrices to Solve Linear Systems

$$2x - 7y = -21$$

$$-x + 4y = 12$$

Step 1: Convert to a matrix equation

$$A \quad X \quad = \quad B$$
$$\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

$$A X = B$$

$$\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

$$A^{-1} A X = A^{-1} B$$

$$X = A^{-1} B$$

Step 2: We need to find A^{-1}

$$A X = B$$

$$\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

$$A^{-1} A X = A^{-1} B$$

$$X = A^{-1} B$$

Step 2: We need to find A^{-1}

Step 3: Find the Inverse Matrix

$$A = \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \rightarrow \left(\begin{array}{cc|cc} 2 & -7 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{array} \right) \quad \begin{array}{l} 1) \text{ Write the matrix and, to its} \\ \text{right, write the identity matrix.} \end{array}$$

2) We need a '1' in the upper left position.

$$\begin{array}{l} R_1 \rightarrow R_2 \\ R_2 \rightarrow R_1 \end{array} \left(\begin{array}{cc|cc} -1 & 4 & 0 & 1 \\ 2 & -7 & 1 & 0 \end{array} \right) \quad R_1 \rightarrow -R_1 \left(\begin{array}{cc|cc} 1 & -4 & 0 & -1 \\ 2 & -7 & 1 & 0 \end{array} \right)$$

3) Perform row operations on the left side matrix until you get zeroes below and then above the main diagonal.

$$\begin{array}{l} R_2 = R_2 - 2R_1 \\ R_1 = R_1 + 4R_2 \end{array} \left(\begin{array}{cc|cc} 1 & -4 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{array} \right) \quad \left(\begin{array}{cc|cc} 1 & 0 & 4 & 7 \\ 0 & 1 & 1 & 2 \end{array} \right) \quad A^{-1} = \begin{bmatrix} 13 & -3 \\ -4 & 1 \end{bmatrix}$$

Step 4: matrix multiplication

$$A^{-1} A X = A^{-1} B$$

$$\begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} (4)(2)+(7)(-1) & (4)(-7)+((7)(4) \\ (1)(2)+(2)(-1) & (1)(2)+(2)(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1} A X = A^{-1} B$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (4)(-21)+(7)(12) \\ (1)(-21)+(2)(12) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$1 \quad X = A^{-1} B$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -21 \\ 12 \end{pmatrix}$$

Your turn: multiply the right side.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$