Math-1050

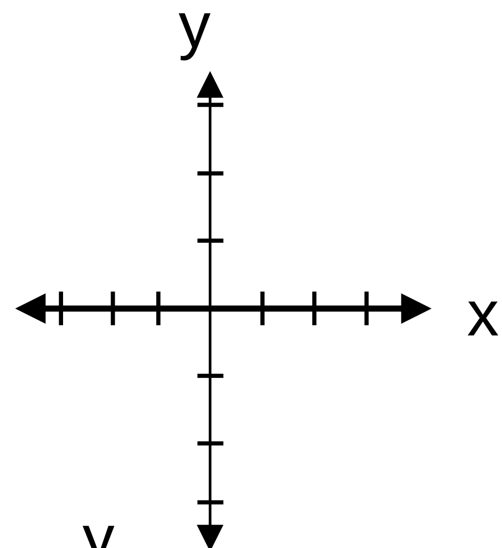
Session 29

Multivariate Linear Systems and Row Operations

When considering space, What is a Dimension?

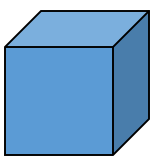
Flat: (x, y coordinate plane)

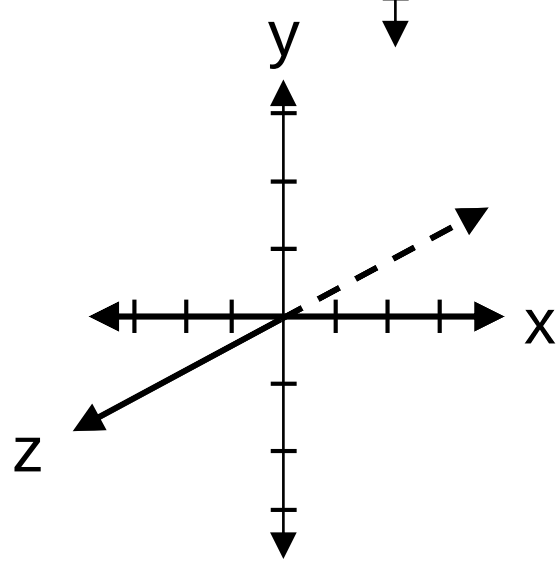
→ 2 Dimensions



Solid: (x, y, z coordinate space)

→ 3 Dimensions





What shape does an Equation Make?

$2x + 3y = 6$ → line (in "2" space)

$2x + 4y + 4z = 12$ → plane (in "3" space)

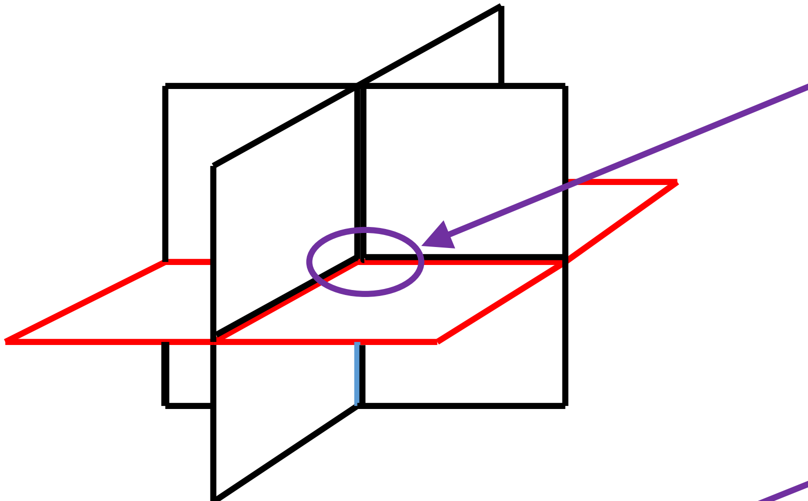
$2x + y - z = 5$

$3x - 2y + z = 16$

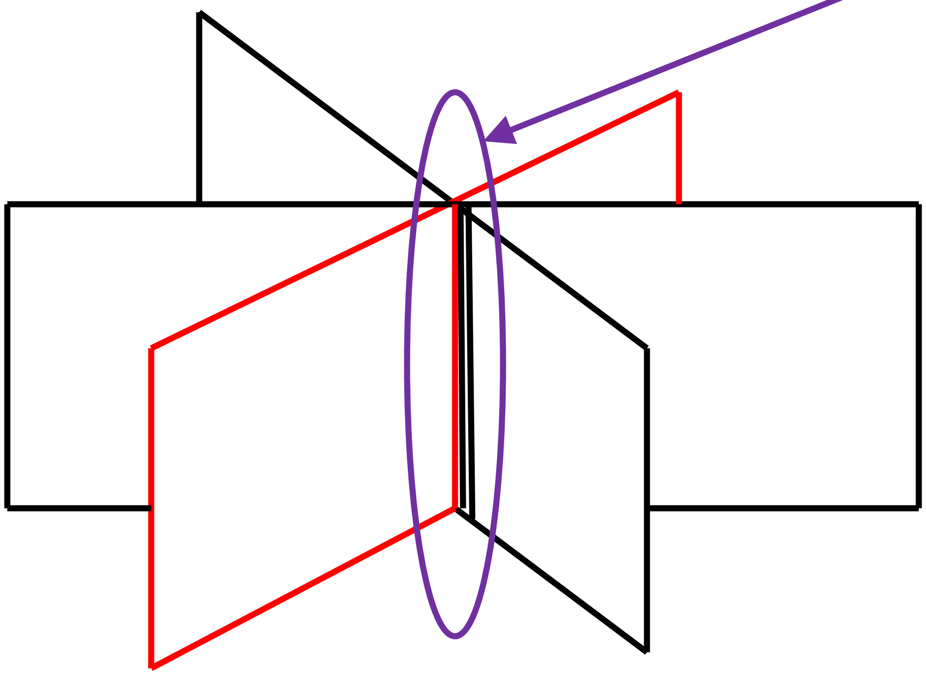
$4x + 3y - 5z = 3$

A system of 3 equations with 3 variables → 3 planes (in "3" space)

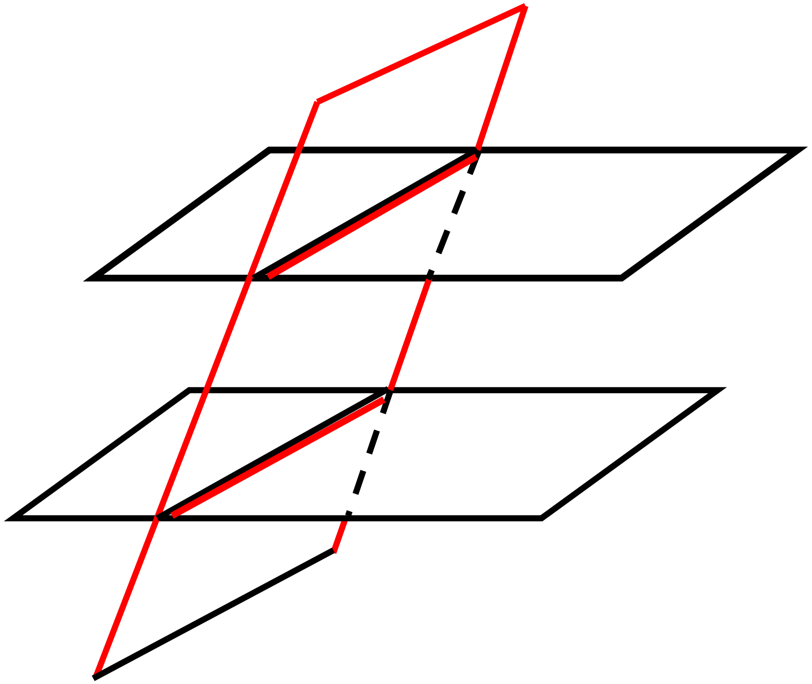
Categories of solutions in “3 space”



Exactly one point, an “ordered triple” (x, y, z) → ***one unique solution***.



(x, y, z) “triples” on a Line: ***INFINITELY many solutions***

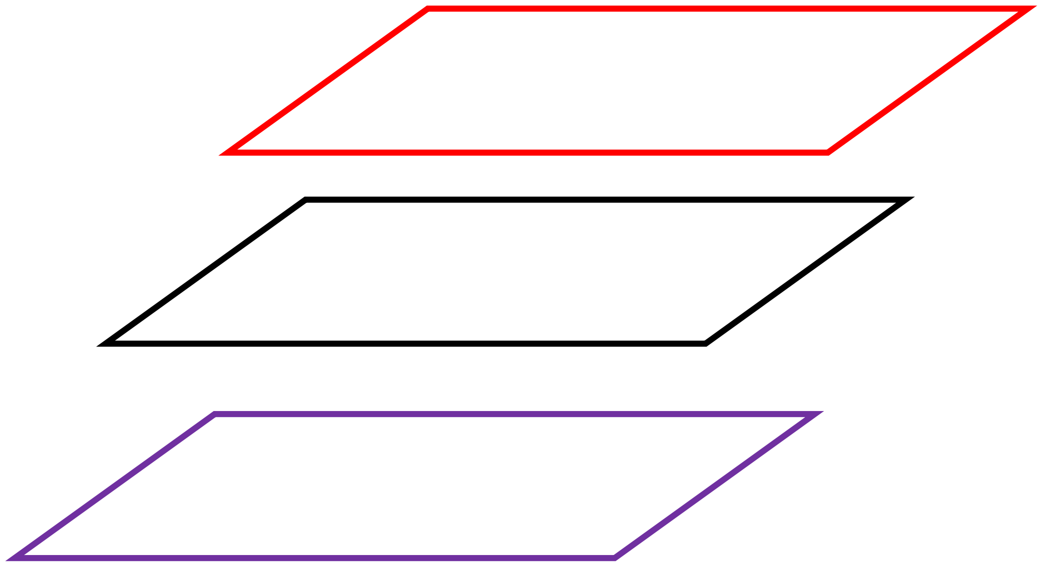


2 parallel planes intersected by a 3rd plane→ no common solution for all 3 planes → *no solution*.

Another possibility: No planes are parallel

→NO POINT SATISFIES ALL 3 EQUATIONS

→No solution



3 parallel planes → No points of intersection → *No solution*

Solving Systems of Linear Equations

- Elimination or Gauassian Elimination
- Gaussian Elimination

$$2x + y - z = 5$$
$$3x - 2y + z = 16$$
$$4x + 3y - 5z = 3$$

Does *rearranging the order* of the equations change the point of intersection?

$$2x + y - z = 5$$
$$3x - 2y + z = 16$$
$$4x + 3y - 5z = 3$$

$$3x - 2y + z = 16$$
$$2x + y - z = 5$$
$$4x + 3y - 5z = 3$$

Does *multiplying one equation by a constant* (on the left side of the equal sign and the right side) change the point of intersection?

Why or why not?

$2x + y - z = 5$

$3x - 2y + z = 16$

$4x + 3y - 5z = 3$

$3x - 2y + z = 16$

$2x + y - z = 5$

$8x + 6y - 10z = 6$

Does *adding the same number or expression to both sides* of an equation change the point of intersection?

$2x + y - z = 5$

$3x - 2y + z = 16$

$4x + 3y - 5z = 3$

$3x - 2y + z = 16$

$2x + y - z = 5$

$5 + 4x + 3y - 5z = 5 + 3$

<u>Equivalent Equations</u> Have the same solution.

The following operations produce an equivalent system of linear equations.

- Interchange any two equations of the system.

- Multiply (or divide) one of the equations by any non-zero real number.

- Add a multiple of one equation to any other equation in the system.

$2x - 3y = 8$

$7x + 2y = 2$

Rewrite as a single matrix (called an *Augmented Matrix*)

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ 7 & 2 & 2 \end{array}\right]$$

We use *row operations* to solve a system of equations written in single-matrix form.

Row Operations Using the Properties of Equality (+/-/x/÷) on a system of equations that has been written as a single matrix, in order to solve the system of equations.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right)$$

We will use repeated "row operations" to convert the matrix into this one

$$\begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ \boxed{z = 2} \end{aligned}$$

Translated back into a system of equations, is…

After row operations

$$\begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Original System of Equations

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Translated to system of equations

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \end{aligned}$$

$z = 2$

(1) plug $z = 2$ into 2nd equation and solve for 'y'

$y + 3(2) = 5$

$y = -1$

(2) plug $z = 2$ and $y = -1$ into 1st equation and solve for 'x'

$x - 2(-1) + 3(2) = 9$

$x = 1$

Row Echelon Form of a Matrix

A matrix is in row echelon form if the following conditions are satisfied.

1. 1's down the main diagonal
2. 0's below the main diagonal
3. Rows consisting entirely of 0's (if there are any) occur at the bottom of the matrix.

$$\begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Elementary Row Operations on a Matrix

A combination of the following operations will transform a matrix to reduced row echelon form.

1. Interchange any two rows.

2. Multiply all elements of a row by a nonzero real number.

3. Add a multiple of one row to any other row.

$$\begin{aligned} x - y + 2z &= -3 \\ 2x + y - z &= 0 \\ -x + 2y - 3z &= 7 \end{aligned}$$

Convert to a single matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 2 & 1 & -1 & 0 \\ -1 & 2 & -3 & 7 \end{array}\right)$$

The right most column are the numbers on the right side of the equal sign.

The vertical line is used by our textbook to that this is an augmented matrix representing a system of linear equations.

Row Operations -- the "big picture"

We are going to rewrite the matrix numerous times. Each rewrite will change one row only. Our objective is to eliminate the x-variables (1st column) below the 1st row like this.

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 2 & 1 & -1 & 0 \\ -1 & 2 & -3 & 7 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -3 & 7 \end{array}\right)$$

Eliminate the y-variables (2nd column) below the 2nd row

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ 0 & 0 & -3 & 7 \end{array}\right)$$

Solve the following system using Row Operations

$$x - y + 2z = -3$$
$$2x + y - z = 0$$
$$-x + 2y - 3z = 7$$

Convert to a matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 2 & 1 & -1 & 0 \\ -1 & 2 & -3 & 7 \end{array}\right)$$

Using elimination, we would multiply the first equation by (-2) then add equations 1 and 2

In row operations we do the same thing, but we replace row 2 with the result of this elimination step.

$$-2(x - y + 2z) = -2(-3)$$
$$2x + y - z = 0$$
$$-2x + 2y - 4z) = 6$$
$$0x + 3y - 5z = 6$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ -1 & 2 & -3 & 7 \end{array}\right)$$

New Row #2 = (-2)Row #1 + Row #2

(1) In row operations we say "Eliminate the x-variable on the second row".

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 2 & 1 & -1 & 0 \\ -1 & 2 & -3 & 7 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ -1 & 2 & -3 & 7 \end{array}\right)$$

$$R_{new2} = (-2)R_{old1} + R_{old2}$$

This short explanation below the matrix will help you to find errors.

$$\begin{array}{rccc|c} & (-2)(1 & -1 & 2) & (-3)(-2) \\ + & 2 & 1 & -1 & 0 \\ \hline & 0 & 3 & -5 & 6 \end{array}$$

(2) Eliminate the x-variable in the 3rd row.

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ -1 & 2 & -3 & 7 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -1 & 4 \end{array}\right)$$

$$R_{new3} = R_{old1} + R_{old3}$$

$$\begin{array}{cccc|c} & 1 & -1 & 2 & -3 \\ + & -1 & 2 & -3 & 7 \\ \hline & 0 & 1 & -1 & 4 \end{array}$$

Only the 3rd row changes!

(3) Eliminate the y-variable on the third row.

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -1 & 4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 1 & 0 & 1 & 1 \end{array}\right)$$

Can we use row 1 to eliminate the y-variable in row 3?

$$R_{new3} = R_{old1} + R_{old3}$$

	1	-1	2	-3
+	0	1	-1	4
	1	0	1	1

Can only use rows 2 and 3 at this point!!! Otherwise, you don't have zero in the 1st position of row 3.

(3) Eliminate the y-variable on the third row.

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -1 & 4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ 0 & 0 & -2 & -6 \end{array}\right)$$

$$R_{new3} = -3 * R_{old3} + R_{old2}$$

$$\begin{array}{rccc|c} & (-3)(0 & 1 & -1) & (-3)(4) \\ + & 0 & 3 & -5 & 6 \\ \hline & 0 & 0 & -2 & -6 \end{array}$$

We have 0's below the main diagonal, we can now solve the system

Translate this to a system of equations

x – y + 2z = -3

3y – 5z = 6

-2z = -6

z = 3

3y – 5(3) = 6

3y = 21

y = 7

x – 7 + 2(3) = -3

x – 1 = -3

x = -2

Classes of Solutions for 3 Equations with 3 unknowns

Gaussian Elimination results in:

- a unique solution (the planes intersect at a point)
- something silly like: $3 = 3$ (Infinitely many solutions)
- Again something silly like: $5 = -9$
  (there are no solutions)

Solve using Gaussian Elimination

$$\begin{aligned} x - y + z &= 0 \\ 2x \quad\; - 3z &= -1 \\ -x - y + 2z &= -1 \end{aligned}$$