Math-1050

Session #27

More Exponential Modeling

Write the following numbers as powers of 2:

$5 \; \boxed{= 2^{2.322}}$ $\qquad 2^x = 5 \qquad \log_2 5 = x \qquad x \approx 2.322$

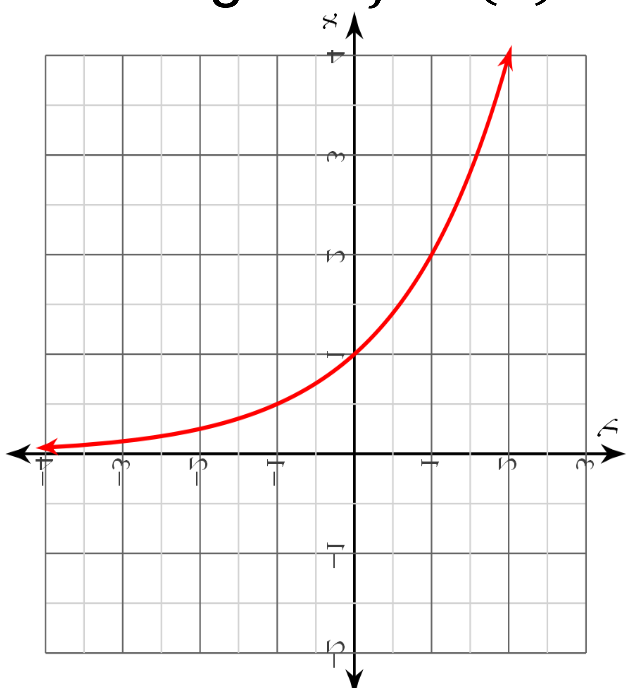
$1.5 \quad = 2^{0.585} \qquad\qquad 0.5 \quad = 2^{-1}$

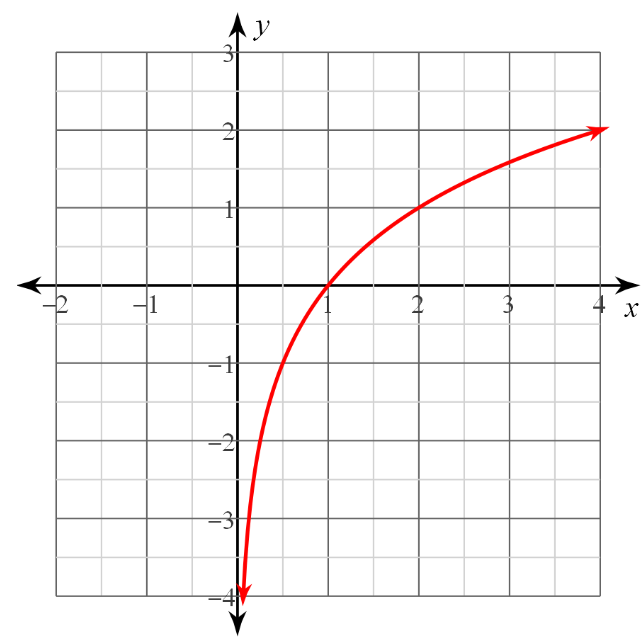
$-2 = 2^x$ Not possible: Why? $log_2(-2) = ?$

$f^{-1}(x) = 2^x \qquad\qquad f(x) = \log_2(x)$

The range of $f^{-1}(x)$ is y >0

The domain of $\mathrm{f}(x)$ is x >0





We can write any number as “e” ……(raised to some exponent).

$$e^1 \approx 2.718$$

$$e^x = 3$$ ‘x’ is between 1 and 2.

$$e^2 \approx 7.389$$

How would you find x?

$$\ln e^x = \ln 3$$

$$x \ln e = \ln 3$$

$$e^{1.099} \approx 3$$

$$x = \ln 3$$

$$x \approx 1.099$$

Rewrite the following numbers as a power of "e"

$2 \qquad e^{x}=2 \qquad x=\ln 2 \qquad e^{0.693}\approx 2$

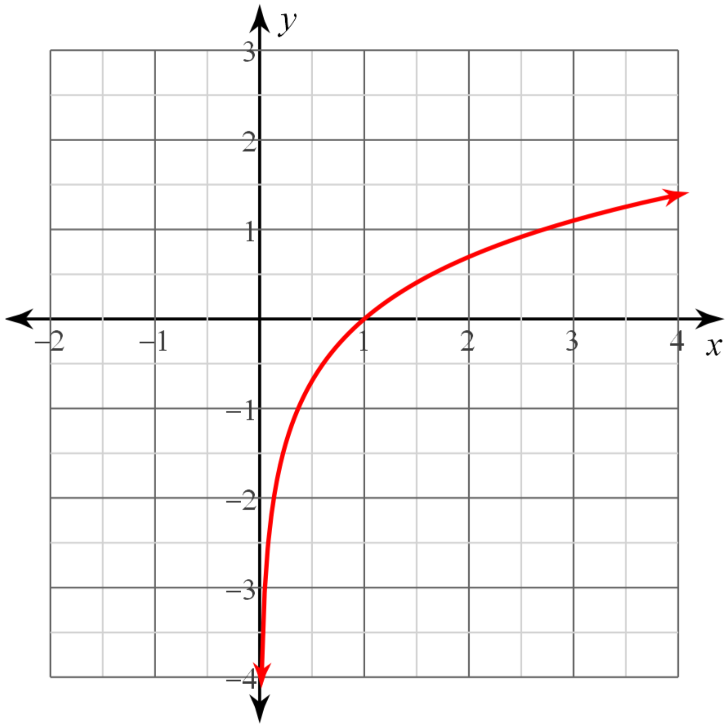
$1.05 \qquad \approx e^{0.049}$

$0.98 \qquad \approx e^{-0.020}$

$0.5 \qquad \approx e^{-0.693}$

-3 Not possible: Why?

$\ln(-3)$ doesn't exist



$y=\ln x$

We can rewrite the base of any exponential as a power of ‘e’.

$$y = 2^x$$

$$y = \left(e^k\right)^x$$

$$e^k = 2$$

$$k = \ln 2$$

$$k \approx 0.693$$

$$y = e^{kx}$$

$$e^{0.693} \approx 2$$

$$y = e^{0.693x}$$

Rewrite the following as base ‘e’ exponential equations.

$$y = 4^x \qquad = e^{1.386x}$$

$$y = 1.1^x \qquad = e^{0.095x}$$

$$y = 1.01^x \qquad = e^{0.010x}$$

$$y = 0.85^x \qquad = e^{-0.163x}$$

$$y = 0.25^x \qquad = e^{-1.386x}$$

How can you distinguish between growth and decay for…

<u>A base “b” exponential?</u>

$$y = b^x$$

$0 < b < 1$	decay
$b > 1$	growth

<u>A base “e” exponential?</u>

$$y = e^{kx}$$

$k < 0$	decay
$k > 0$	growth

Rate: the change of one quantity compared to the change in another quantity using a fraction.

In pure mathematics we would call this a slope.

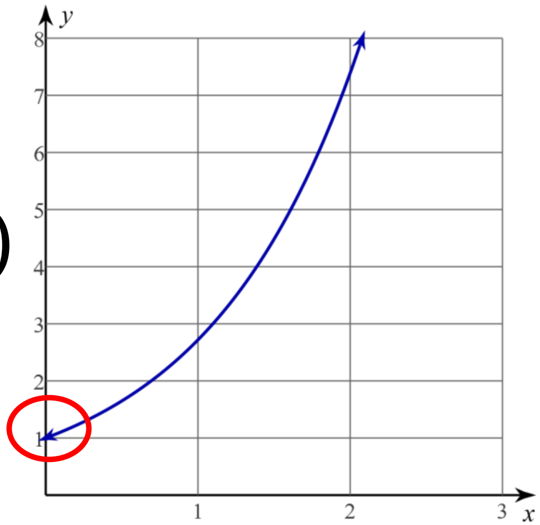
$$\frac{\Delta y}{\Delta x} = slope$$

Rate: (a ratio of quantities) becomes a new quantity.

$$\frac{\Delta\ \text{temp}}{\Delta\ \text{time}} = \text{heatup/cooldown rate}$$

Many natural processes follow the model given by:

$$A(t) = A_0 e^{kt}$$



$A(t)$

A_0

t

"A(t)" is the future amount (as a function of time)

"A_0" is initial amount (the amount at time = 0)

"k" is the growth (or decay) rate

→ K > 0 → "uninhibited growth"

→ K < 0 → "decay"

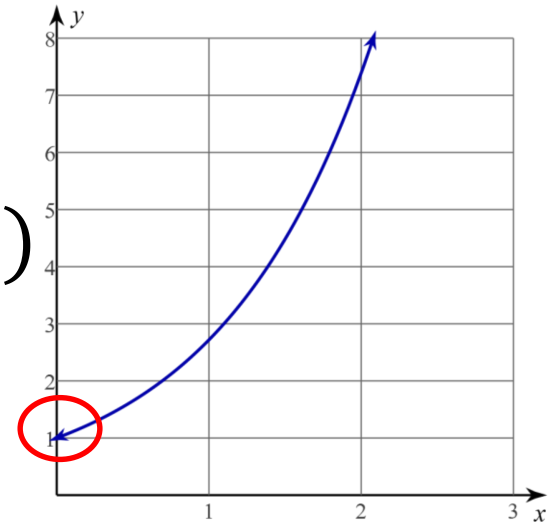
"t" is the time from some reference point (stopwatch time)

$$A(t) = A_0 e^{kt}$$

Bacteria Growth

$$N(t) = N_0 e^{kt}$$

$N(t)$



N_0

t

“N(t)” is the number of bacteria in the colony (as a function of time)

"N_0" is initial number of bacteria in the colony (the amount at time = 0)

“k” is the growth (or decay) rate

“t” is the time since the initial population size was measured (stopwatch time)

Bacteria Growth

$$N(t) = N_0 e^{kt}$$

The population of bacteria in a colony can be modeled by:

$$N(t) = 100e^{0.045t}$$

Where the units of 'N' are grams of bacterial and time is measured in days.

What is the initial population?

100 grams

What is growth rate (in %)?

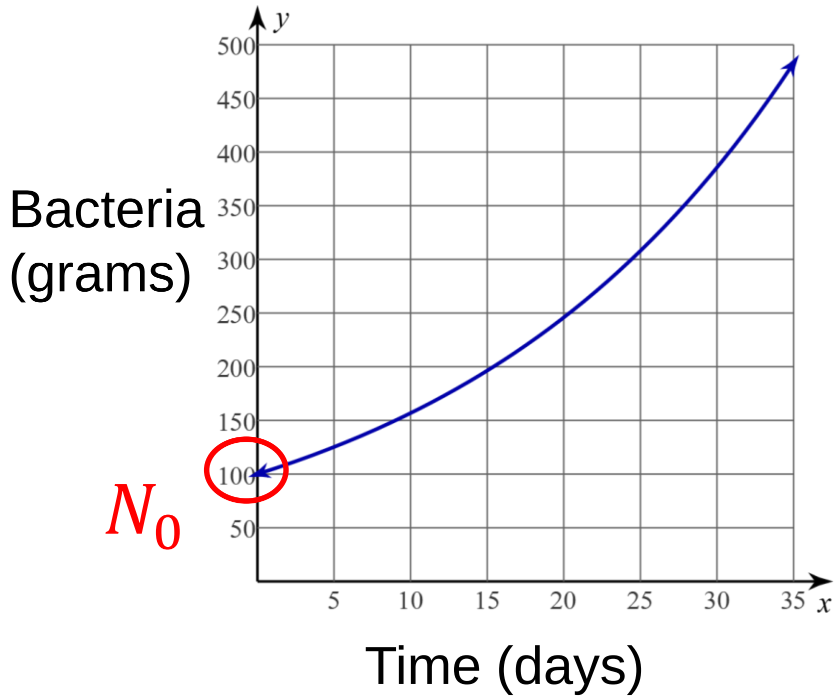
4.5%

What is the population in: 10 days, 20 days, 30 days?

(N, t) in units of (grams, days):
(10, 156.8), (20, 246.0), (30, 385.7)

Bacteria Growth

$$N(t) = 100e^{0.045t}$$



Draw and correctly label a graph showing the model.

If the colony was eradicated after 35 days, what is the domain of the relation?

D(N) (in days) = [0, 35]

What is the range of the relation?

D(N) (in grams) = [100, 483.1]

Bacteria Growth

$$N(t) = 100e^{0.045t}$$

How long does it take the colony to reach 400 grams?

$$400 = 100e^{0.045t}$$

$$\div 100 \qquad \div 100$$

$$4 = e^{0.045t}$$

$$ln4 = 0.045\text{t}$$

$$\div 0.045 \qquad \div 0.045$$

$$t = \frac{ln4}{0.045} = 30.8 \text{ days}$$

How long does it take the colony to double in size?

$$200 = 100e^{0.045t}$$

$$\div 100 \qquad \div 100$$

$$2 = e^{0.045t}$$

$$ln2 = 0.045\text{t}$$

$$\div 0.045 \qquad \div 0.045$$

$$t = \frac{ln2}{0.045} = 15.4 \text{ days}$$

The “half life” of Carbon-14 (a radioactive isotope of carbon), is 5730 years. Calculate the decay rate for carbon-14. The decay rate is the “k” of the exponent of ‘e’.

$$A(t) = A_0 e^{kt}$$

$$0.5A_0 = A_0 e^{k(5730)}$$

$$0.5 = e^{5730(k)}$$

$$\ln 0.5 = 5730k$$

$$k = {}^{-0.00012}/_{yr}$$

The "decay rate" of Carbon-14 (a radioactive isotope of carbon), is -0.00012. If there were 5 grams of C-14 initially, how many grams will be present after 15,000 years?

$$A(t) = A_0 e^{kt} \qquad k = -0.00012$$

$$A(15{,}000) = 5e^{-0.00012(15{,}000)}$$

$$A(15{,}000) = 0.83 \text{ gms}$$

If the amount of carbon 14 in some wood was found to be 1.5% of the original amount, how old is the wood.?

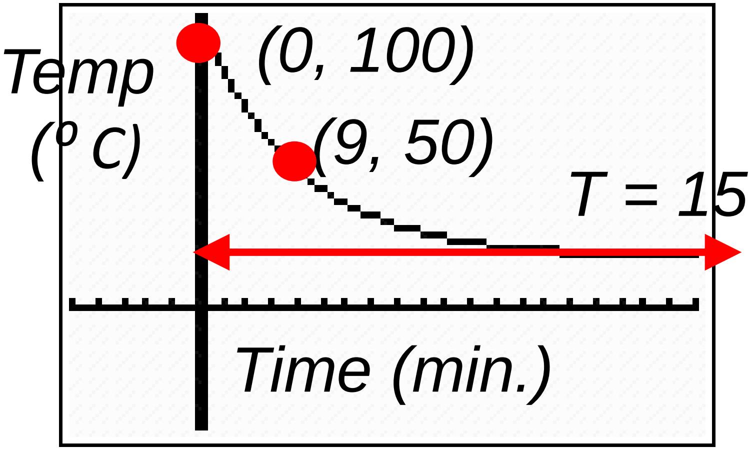
$$\% = \frac{Part}{Whole}$$

$$0.015 = e^{-0.00012*t}$$

$$\ln(0.015) = -0.00012t$$

$$t = \frac{ln(0.015)}{-0.00012}$$

$$= 34{,}997.5\ yrs$$



Boiling water (100º C) is taken off the stove to cool in a room at 15º C. After 9 minutes, the water's temperature is 50 C.

Write the modeling equation as a base 'b' exponential.

$$T(t) = a(b)^t + k$$

1) Horizontal Asymptote

$$T(t) = a(b)^t + 15$$

2) y-intercept

$$100 = a(b)^0 + 15$$

$$a = 85$$

3) "nice point"

$$50 = 85(b)^9 + 15$$

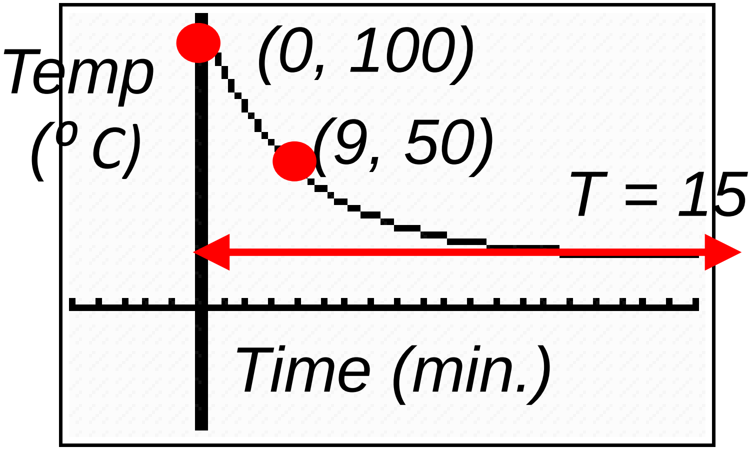
$$\left(\frac{50-15}{85}\right) = (b)^9$$

$$\left(\frac{50-15}{85}\right)^{1/9} = b$$

$$b = 0.906$$

4) Final equation

$$T(t) = 85(0.906)^t + 15$$



Boiling water (100º C) is taken off the stove to cool in a room at 15º C. After 9 minutes, the water's temperature is 50 C.

Write the modeling equation as a base 'e' exponential.

$$T(t) = 85(0.906)^t + 15$$

$$e^k = 0.906$$

$$\ln 0.906 = k$$

$$k = -0.987$$

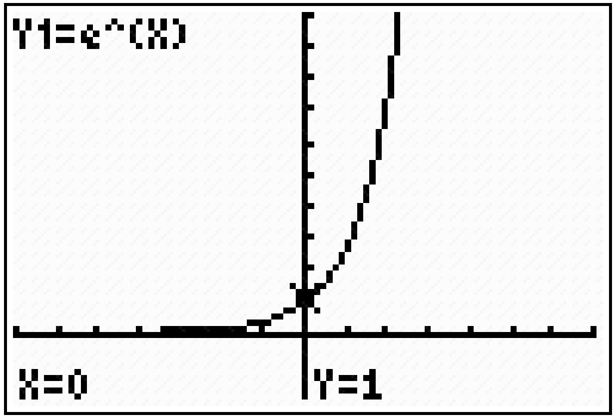
$$T(t) = 85e^{-0.987t} + 15$$

$$A(t) = A_0 e^{kt}$$

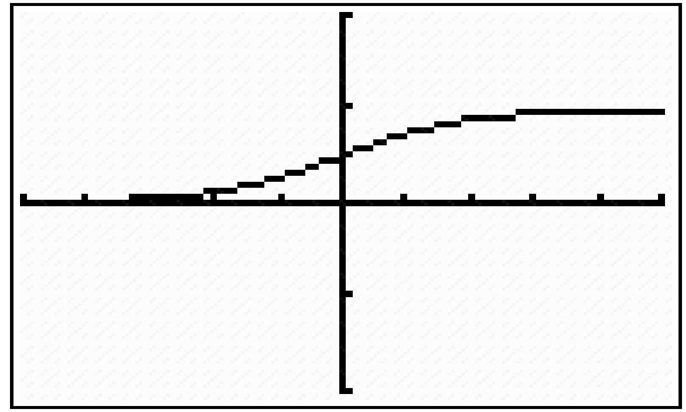
A hard-boiled egg at temperature 212^{o} F is placed in 60^{o} F water to cool. 5 minutes later the temperature of the egg is 95^{o} F. When will the egg be 75^{o}C?

A cake taken out of the oven at temperature of 350^{o} F. It is placed on in a room with an ambient temperature of 70^{o}F to cool. Ten minutes later the temperature of the cake is 150^{o}F. When will the cake be cool enough to put the frosting on (90^{o}F) ?

Which of the two models best represents population as a function of time (bacteria, zebras, monkeys, etc.) in the real world?



Or



Exponential

Logistic

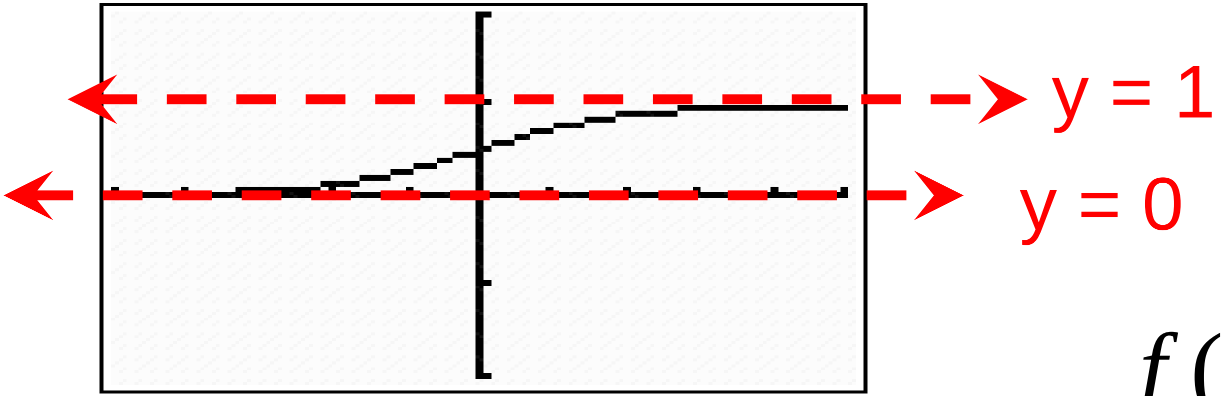
Maximum Sustainable Population

Exponential growth is unrestricted, but population growth often is not. For many populations, the growth begins exponentially, but eventually slows and approaches a limit to growth called the **maximum sustainable population**.

We must use Logistic function if the growth is limited !!!

What factors can limit the size of the population?

Logistic Function



$$f(x) = \frac{1}{1+e^{-x}}$$

$$f(x) = \frac{1}{1+e^{-x}}$$

$$x \to -\infty \quad y \to ?$$

$$\text{limit}_{x\to-\infty} f(x) = ?$$

$$f(x) = \frac{1}{1+e^{x}}$$ huge

$$x \to -\infty \quad y \to 0$$

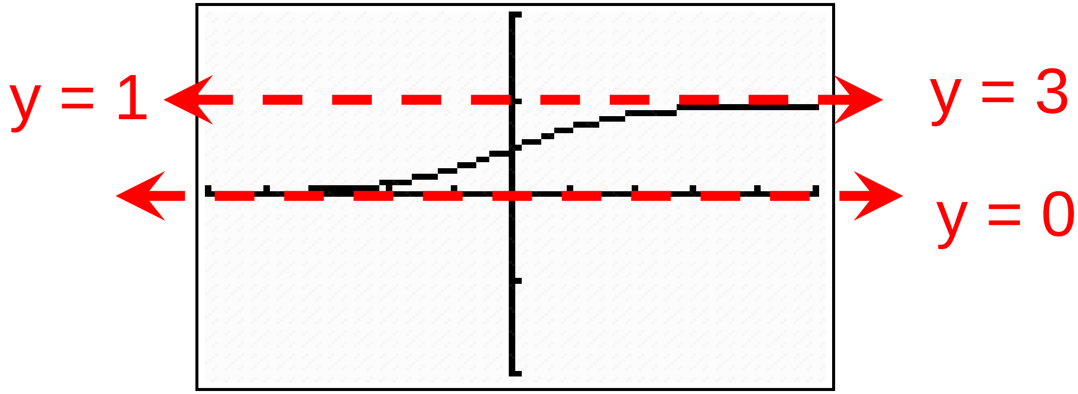
$$f(x) = \frac{1}{1+\left(\frac{1}{e}\right)^{x}}$$ 0

$$x \to \infty \quad y \to ?$$

$$\text{limit}_{x\to\infty} f(x) = ?$$

$$f(x) = \frac{1}{1+0}$$

Logistic Function



Parent Function:

$$f(x) = \frac{1}{1+e^{-x}}$$

Vertical stretch by a factor of '3'

$$g(x) = \frac{3}{1+e^{-x}}$$

What does vertically stretched by a factor of '3' mean?

y-values of the parent function are multiplied by 3.

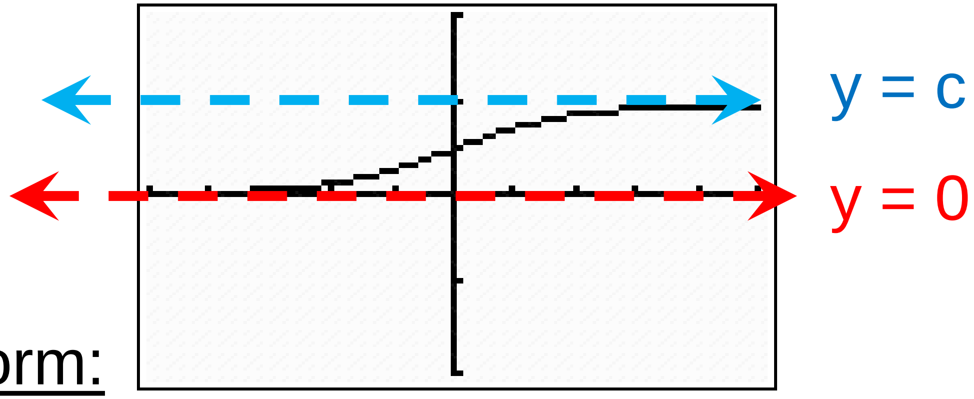
What happens to the horizontal asymptotes?

y = 0 stays y = 0.

y = 1 becomes y = 3

Parent Function:

$$f(x)=\frac{1}{1+e^{-x}}$$



General Transformation Form:

$$f(x)=\frac{c}{1+ae^{-kx}}$$

'c' is the "limit to growth"
'k' is percent rate change
'a' is a constant

Logistic Growth:

k > 0

Example: "limit to growth" = 12 and % rate of change:
k = 200% = 2

$$f(x)=\frac{12}{1+e^{-2x}}$$

Logistic Decay:

k < 0

Example: "c" = 12 and % rate of change:
k = -50% = -0.5

$$f(x)=\frac{12}{1+e^{0.5x}}$$

Is it logistic growth or decay?

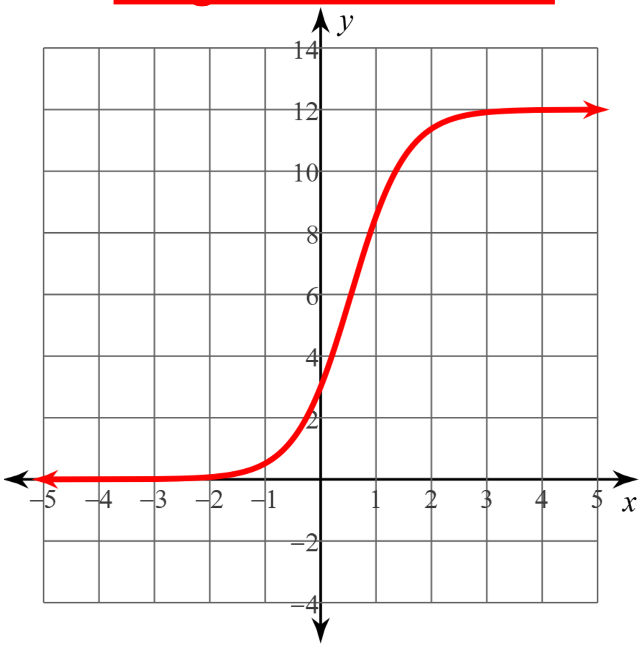
$$f(x)=\frac{c}{1+ae^{-kx}}$$

$$f(x)=\frac{12}{1+4e^{-2x}}$$

k = 200% = 2

k > 0

<u>Logistic Growth</u>

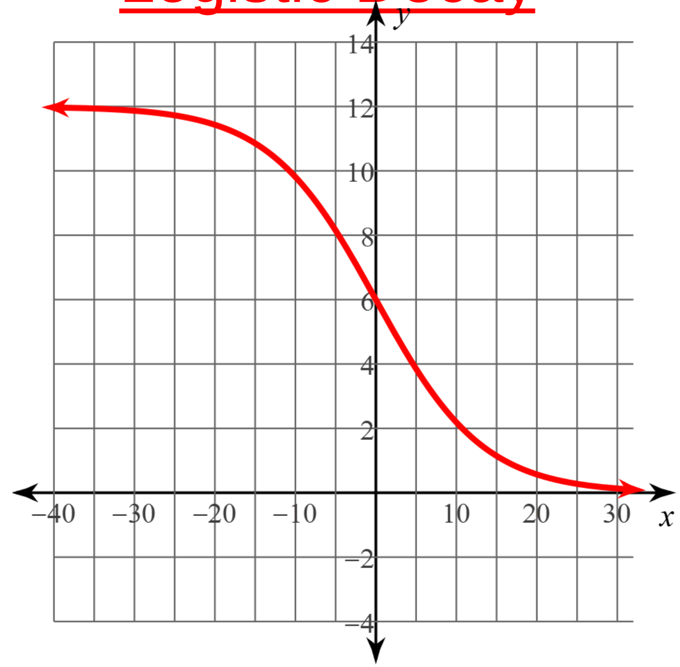


$$f(x)=\frac{12}{1+5e^{0.15x}}$$

k = -15% = -0.15

k < 0

<u>Logistic Decay</u>



A population of fruit flies are placed in a gallon-sized milk jug with banana slices (for food). The population can be modeled with the equation where 't' is in units of 'days'.

$$f(x) = \frac{c}{1 + ae^{-kx}} \qquad P(t) = \frac{230}{1 + 56.5e^{-0.37t}}$$

a) Find the initial population. $P(0) = ?$

b) Find the carrying capacity of the environment. $c = ?$

c) Find percent rate of change. $k = ?$

d) Find the population after 5 days. $P(5) = ?$

e) Find the time for the population to reach one half of the carrying capacity. $P(t) = 115, \quad t = ?$