Math-1050
Session #26

6.6: Solve Exponential and Logarithmic Equations

And

6.7: Compound Interest

The easiest Exponential Equation

$$2^x = 2^{4-x}$$

Exponents have to be equal to each other!

$$x = 4 - x$$

+x +x

$$2x = 4$$

÷x ÷x

$$\boxed{x = 2}$$

Check your answer!

$$2^2 = 2^{4-2}$$

$$7^{2x+1} = 7^{13-4x}$$

$$2x + 1 = 13 - 4x$$

+4x +4x

$$6x + 1 = 13$$

-1 -1

$$6x = 12$$

$$\boxed{x = 2}$$

Solving using "convert to same base"

$$2^{4x-1} = 8^{x-1}$$

"convert to same base"

$$2^{4x-1} = (2^3)^{x-1}$$

Exponent of a power
Exponent Property

$$2^{4x-1} = 2^{3x-3}$$

$$\boxed{x = -2}$$

$$4x - 1 = 3x - 3$$

-3x -3x

$$x - 1 = -3$$

+1 +1

Check your answer!

$$2^{4(-2)-1} = 8^{-2-1}$$

$$2^{-9} = 8^{-3}$$

$$(\ 2^{-9} = 8^{-3}\ )^{-1}$$

$$2^9 = 8^3$$

$$512 = 512$$

Sometimes you can't rewrite the exponentials with the same bases so you have no choice. Use log of a power property.

$$5^x = 7^{2x-1}$$

$$\ln 5^x = \ln 7^{2x-1}$$

$$x \ln 5 = (2x-1)\ln 7$$

÷ ln 5 ÷ ln 5

$$x = (2x-1)\frac{\ln 7}{\ln 5}$$

$$x = (2x-1)(1.21)$$

$$x = 2.42x - 1.21$$

+1.21 +1.21

$$1.21 + x = 2.42x$$

-x -x

$$1.21 = 1.42x$$

÷ 1.42 ÷ 1.42

$$\boxed{x = 0.85}$$

Instead of Using the Log of a Power Property Using Log base ‘e’, you can use log base 5.

$$5^x = 7^{2x-1}$$

$$log_5 5^x = log_5 7^{2x-1}$$

$$x \log_5 5 = (2x-1)\log_5 7$$

$$x = (2x-1)1.20906$$

$$x = (2x-1)(1.21)$$

$$x = 2.42x - 1.21$$

+1.21 +1.21

$$1.21 + x = 2.42x$$

-x -x

$$1.21 = 1.42x$$

÷ 1.42 ÷ 1.42

$$\boxed{x = 0.85}$$

Instead of Using the Log of a Power Property Using Log base ‘5’, you can use log base 7.

$$5^x = 7^{2x-1}$$

$$log_7 5^x = log_7 7^{2x-1}$$

$$x \log_7 5 = (2x - 1) \log_7 7$$

$$0.82709x = 2x - 1$$

+1 +1

-0.83x -0.83x

$$1 = 1.17x$$

$$1 = 1.17x$$

÷ 1.17 ÷ 1.17

$$\boxed{x = 0.85}$$

Sometimes the questions will ask for the *EXACT SOLUTION*.

$$5^x = 7^{2x-1}$$

$$log_7 5^x = log_7 7^{2x-1}$$

$$x \log_7 5 = (2x-1)\log_7 7$$

$$x \log_7 5 = 2x - 1$$

$+1$ $+1$

$$-x \log_7 5 = -xlog_7 5$$

$$1 = 2x - xlog_7 5$$

$$1 = x(2 - log_7 5)$$

$\div (2 - \log_7 5)$ $\div (2 - log_7 5)$

$$x = \frac{1}{(2 - log_7 5)}$$

Solving using "log of a power" property

$$8^{x+2} = 4^{x-2}$$

Take natural log of both side

$$\ln 8^{x+2} = \ln 4^{x-2}$$

"log of pwr property"

$$(x+2)\ln 8 = (x-2)\ln 4$$

÷ ln 8 ÷ ln 8

$$(x+2) = (x-2)\frac{\ln 4}{\ln 8}$$

simplify

$$x+2 = (x-2)(0.6666666)$$

simplify

$$x+2 = (x-2)\left(\frac{2}{3}\right)$$

$$3(x+2) = (x-2)\left(\frac{2}{3}\right)(3)$$

$$3x+6 = 2x-4$$

-2x - 6 -2x - 6

$$\boxed{x = -10}$$

Solve using the "log of a power property"

$$5^{x+2} = 4^{x-2}$$ Take natural log of both sides

$$\ln 5^{x+2} = \ln 4^{x-2}$$ Power property

$$(x+2)\ln 5 = (x-2)\ln 4$$

÷ ln 5 ln 5

$$x+2 = (x-2)\frac{\ln 4}{\ln 5}$$

$$x+2 = (x-2)(0.8614)$$

$$x+2 = 0.8614x - 1.7227$$

-x **-x**

$$2 = -0.1387x - 1.7227$$

+1.7227 +1.7227

$$3.7227 = -0.1387x$$

÷ -0.1387 ÷ -0.1387

$$\boxed{x = -26.8399}$$

When the unknown value is in the exponent, always remember "a log is an exponent"

$$8^x - 2 = 5$$ "Isolate the exponential"

+2 +2

$$8^x = 7$$ "convert to a log"

$$x = \log_8 7$$

$$\boxed{x = 0.9358}$$

Solve using "undo the exponential"

$$3^{2x-1}+5=7$$ "Isolate the exponential"

-5 -5

$$3^{2x-1}=2$$ "Undo the exponential"

$$2x-1=\log_3 2$$

→ Change of base formula

$$2x-1=\frac{\ln 2}{\ln 3}$$

$$2x-1=0.63093$$

+1 +1

$$2x=1.63093$$

÷2 ÷2

$$\boxed{x=0.815}$$

$$2x-1=0.63093$$

+1 +1

$$2x=1.63093$$

÷2 ÷2

$$\boxed{x=0.815}$$

"Quadratic Form": a trinomial that looks like a standard form quadratic of the form $y = ax^{2n} + bx^{n} + c$

$y = x^4 + 3x^2 + 2$

Use "m-substitution"

$x^4 = m^2, x^2 = m^1$

$y = m^2 + 3m^1 + 2$

Factor

$y = (m + 2)(m + 1)$

Use "m-substitution"

$y = (x^2 + 2)(x^2 + 1)$

Zeroes: $x = i\sqrt{2}, -i\sqrt{2}, i, -i$

Extraneous solution: an apparent solution that does not work when plugged back into the original equation.

Square root equations Radicands cannot be negative

$$5 = \sqrt{2x - 3}$$

Domain of *possible solutions:* $2x - 3 \geq 0$ $x \geq \frac{3}{2}$

$5 = \sqrt{2x - 3}$ $25 = 2x - 3$ $28 = 2x$ $x = 14$

Apparent solution (x = 14) is *in the domain of possible solutions*

x = 14 is not an *extraneous solution.*

Log equations Logarands must be positive

$$5 = \log_3(2x - 3)$$

Domain of *possible solutions:* $2x - 3 > 0$ $x \geq \frac{3}{2}$

$5 = \log_3(2x - 3)$ "A log is an exponent."

$3^5 = 2x - 3$ $243 = 2x - 3$ $246 = 2x$ $x = 123$

Apparent solution (x = 123) is *in the domain of possible solutions*

x = 123 is not an *extraneous solution.*

"Quadratic Form": a trinomial that looks like a standard form quadratic of the form $y = ax^{2n} + bx^n + c$

$y = 3^{2x} + 3^x - 2$ Use "m-substitution" $m = 3^x$

$y = (3^x)^2 + 3^x - 2$ $y = m^2 + m - 2$ Factor

$0 = (m+2)(m-1)$ Find the zeroes

$m = -2, 1$ Use "m-substitution" $m = 3^x$

$3^x = -2$ $3^x = 1$ "A log is an exponent"

$x = \log_3(-2)$ $x = \log_3(1)$

Domain of log function: x > 0

$\boxed{x = 0}$

Extraneous solution

The easiest log equation. $\log(x+3) = \log(2x-1)$

$x + 3 = 2x - 1$ $\boxed{\rightarrow x = 4}$

Domain of *possible solutions* is the *more restictive of:*

$x + 3 > 0$ or $2x - 1 > 0$

$x > -3$ $x > \frac{1}{2}$

Apparent solution (x = 2) is *in the domain of possible solutions*

x = 4 is not an *extraneous solution.*

$$\log_2 5^x = 5$$

Power property of logarithms

$$x \log_2 5 = 5$$

→ Change of base

$$x \frac{\ln 5}{\ln 2} = 5$$

$$2.32192x = 5$$

÷2.32192 ÷2.32192

$$\boxed{x = 2.1534}$$

→ Approximate solution

Use inverse property of multiplication

$$\boxed{x = 5 \frac{\ln 2}{\ln 5}}$$

$$\log_3 4^{5x} = 6$$

→ Exact solution

Solving Logs requiring condensing the product.

$\log 2x + \log(x-5) = 2$ "condense the product"

$\log 2x(x-5) = 2$ "undo the logarithm"

$10^2 = 2x(x-5)$ Quadratic → put in standard form

$100 = 2x^2 - 10x$

$2x^2 - 10x - 100 = 0$ Divide both sides by '2'

$x^2 - 5x - 50 = 0$ factor

$(x-10)(x+5) = 0$ Zero product property

$\boxed{x = 10, -5}$

Check for extraneous solutions: $\boxed{x = 10, \cancel{-5}}$

$$\log 2x + \log(x-5) = 2$$

$$\log(2 * 10) + log(10 - 5) = 2$$

$$\log(20) + log(5) = 2$$ All logarands are positive ☺

$$log100 = 2$$ “Condense the product” $$10^2 = 100$$

Checks

$$\log(2)(-5) + log(-5 - 5) = 2$$

$$\log(-10) + log(-10) = 2$$ Negative logarands ☹

x = 5 is an extraneous solution.

More complicated Logarithmic Equations

$$2 + \log_2 5^{x-2} = 7$$

"Isolate the logarithm"

-2 -2

$$\log_2 5^{x-2} = 5$$

"undo the logarithm"

$$(x-2)\log_2 5 = 5$$

$\div \log_2 5$ $\div \log_2 5$

$$x - 2 = 2.15338$$

$+2$ $+2$

Add '2' to both sides.

$$x = 4.1524$$

$$\log_4(5x-1)=3$$

"Isolate the logarithm"

$$5x-1=4^3$$

"undo the logarithm"

5x – 1 = 64

Add '1' to both sides

$$5x=65$$

Divide both sides by '5'

x = 13

Plug back in to check!

$$\log_4(5*13-1)=3$$

$$\log_4 64=3$$

Checks

Exponential Data: <u>what is the equation</u>?

x	f(x)
-2	0.0625
-1	0.25
0	1
1	4
2	16
3	64
4	256

4

4

<u>Integer x-values</u> increment by <u>one</u> each time

y-values increment by the same <u>factor</u> each time.

This number is the “growth factor”

“growth factor” is the <u>base of the exponential.</u>

$$f(x) = 4^x$$

$$f(x) = (1+3)^x$$

Another way to think about it.

$$f(x) = 4^x$$

x	f(x)
0	1
1	4
2	16
3	64
4	256

4

$(1+3)$

"growth factor" is the base of the exponential.

$$y_{next} = y_{previous} * Growth\ factor$$

The "next y-value" equals the "previous y-value" *plus some growth*.

$$16 = 4 + 12$$

$$16 = 4(1+3)$$

growth factor = 1 → no growth

% rate of change

$$f(x) = (1+3)^x$$

You deposit \$100 money into an account that pays 3.5% interest per year. The "rent" is "paid" yearly. How much money will be in the account at the end of the 1st year?

$$A(1) = 100(1 + 0.035)^{(1)} \quad A(1) = 100(1.035)^{(1)}$$

$$A(1) = \$103.5$$

How much will be in the account after the 2nd year?

$$A(2) = A(1)(1.035)^{(1)}$$

$$A(2) = 103.5(1.035)^{(1)}$$

$$A(2) = 100(1.035)^{(1)}(1.035)^{(1)}$$

$$A(t) = A_0(1 + r)^t$$

$$A(2) = 100(1.035)^{(2)}$$

$$A(2) = \$107.12$$

You deposit $100 money into an account that pays 2% interest per year. But the "rent" is paid "monthly." What is the interest rate that is paid each month?

$$\frac{2\%}{year} * \frac{1\text{ year}}{12\text{ months}} = \frac{0.17\%}{month} = \frac{0.0017}{month}$$

How much money will be in the account after 5 months?

$$A(t) = A_0(1+r)^t$$

$$A(5) = 100(1+0.0017)^5$$ Time uses units of <u>months</u>

$$A(5\text{ months}) = \$100.85$$

How much money will be in the account after 7 years?

$$A(7\text{ years}) = 100(1+0.0029)^{12(7)}$$

Time uses units of <u>years</u>.

$$A(t\text{ in yrs}) = A_0(1+{}^{r}\!/_{n})^{nt}$$

$$A(7) = \$127.54$$

"*n*": number of times "rent" is paid per year

<u>Compound interest</u>: the interest (rent) that is paid at the end of period of time.

$$A(t \text{ in yrs}) = A_0(1 + r/n)^{nt}$$

Compounded annually: *"n" = ?* $A(t \text{ in yrs}) = A_0(1 + r/1)^{1*t}$

Compounded semi-annually: *"n" = ?* $A(t \text{ in yrs}) = A_0(1 + r/2)^{2*t}$

Compounded quarterly: *"n" = ?* $A(t \text{ in yrs}) = A_0(1 + r/4)^{4*t}$

Compounded monthly: *"n" = ?* $A(t \text{ in yrs}) = A_0(1 + r/12)^{12*t}$

Compounded weekly: *"n" = ?* $A(t \text{ in yrs}) = A_0(1 + r/52)^{52*t}$

Compounded daily: *"n" = ?* $A(t \text{ in yrs}) = A_0(1 + r/365)^{365*t}$

Compounded hourly: *"n" = ?* $A(t \text{ in yrs}) = A_0(1 + r/8760)^{8760*t}$

Compounded minutely: *"n" = ?* $A(t \text{ in yrs}) = A_0(1 + r/525600)^{525600*t}$

You deposit \$100 money into an account that pays 3% interest per year. The interest is "compounded" monthly. How much money will be in the account at the end of the 5th year?

$$A(t) = A_0(1 + r/k)^{kt} \qquad A(5) = 100(1 + 0.03/12)^{12(5)}$$

$$A(5) = \$116.16$$

What is the doubling time for this account?

$$200 = 100(1 + 0.035/12)^{12t}$$

$$2 = (1.0029)^{12t}$$

$$\log_{1.0029}(2) = 12t \qquad 239.4 = 12t \qquad t = 19.9 \text{ yrs}$$

Continuous compounding

$$A(t) = A_0(1 + {}^{r}/_{k})^{kt}$$

$$A(t) = A_0 e^{rt}$$

$$\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^{n} = \mathrm{e}$$

A bank compounds interest continuously. The annual interest rate is 5.5%. How long would it take for the money in the account to triple?

$$3A_0 = A_0 e^{0.055t}$$

$$3 = e^{0.055t}$$

$$\ln 3 = 0.055t$$

$$t = 19.97 \text{ yrs}$$

Present Value:

"Present value of "A" dollars to be received at 't' years, annual interest rate 'r', compounded 'n' times per year."

→ The original investment amount is the unknown value.

$$A(t) = A_0 e^{rt} \qquad A(t) = A_0 (1 + {}^{r}/_{k})^{kt}$$

Future value.

$$A_0 = A(t) * e^{-rt} \qquad A_0 = A(t) * (1 + {}^{r}/_{k})^{-kt}$$

Effective Rate of Return

The equivalent simple interest rate (compounding period = 1 yr) of an investment that has a different annual interest rate and is compounded more than once per year.

→ The effective rate of return is the unknown value.

$$A_0\left(1 + r_{eff}\right)^t = A_0\left(1 + {r_{new}}/{k}\right)^{kt}$$

Assume 1 year and the same investment in each account

$$1 + r_{eff} = \left(1 + {r_{new}}/{k}\right)^k$$

$$r_{eff} = \left(1 + {r_{new}}/{k}\right)^k - 1$$