Math-1050
Session #23

Exponential Function

Power: An expression formed by repeated multiplication of the same factor.

$$3x^4$$

Coefficient Base Exponent

$$3x^4 \rightarrow 3 * x^4 \rightarrow 3 * x * x * x * x$$

Base: the number that is repeatedly multiplied

Exponent: the number of times the base is used as a factor

The exponent applies to the number or variable immediately to its left, not to the coefficient !!!

Power Function: A single-term equation whose base is the input variable.

$$f(x) = 3x^4$$

Properties of Exponents

Multiply Powers Property: when you multiply same-based powers, you add the exponents.

$$(x^2)(x^3) = (x * x)(x * x * x) = x^5$$

$$x^2 x^3 = x^{2+3} = x^5$$

Exponent of a Power Property: a power (base and an exponent) that has another exponent $(x^2)^3$ is simplified by multiplying the exponents

$$(x^2)^3 = (x * x)(x * x)(x * x) = x^6$$

$$(x^2)^3 = x^{2*3} = x^6$$

Properties of Exponents

Exponent of a Product Property: (an exponent of two or more different-based powers that are being multiplied together) is simplified by multiplying the exponent outside of the parentheses by each of the exponents inside of the parentheses.

$$(xy^3)^2 = (xy^3)(xy^3) = xxy^3y^3 = x^2y^6$$

$$(x^1y^3)^2 = x^{1*2}y^{3*2} = x^2y^6$$

This makes it seem that there is a "distributive property of exponents" → ***there is NOT.***

$$(x+y)^2 \neq x^2 + y^2 \qquad (x+y)^2 = x^2 + 2xy + y^2$$

Properties of Exponents

Negative Exponent Property

"Grab and drag"

$$x^{-2} = \frac{1 * x^{-2}}{1} = \frac{1 * x^{-2}}{1} = \frac{1}{x^2}$$

When you "Grab and drag" the base and its exponent across the "boundary line" between numerator and denominator, you just change the sign of the exponent.

Zero Exponent Property

$10^3 = 1000$ $10^2 = 100$ $10^1 = 10$ $10^0 = 1$

Any base raised to the zero power simplifies to one.

The "Parent" Exponential Function

$$y = b^x$$

exponent

base

$$y = 2^x$$ **(base 2 exponential function)**

$$y = 3^x$$ **(base 3 exponential function)**

$$y = \left(\frac{1}{2}\right)^{\mathrm{x}}$$ **(base 1/2 exponential function)**

The base MUST BE positive and CANNOT equal 1.

$$b = (0,1) \cup (1,\infty)$$

Fill in the output values of the table and graph the points.

$$f(x) = 2^x$$

<u>Growth Factor</u> is the base of the exponential

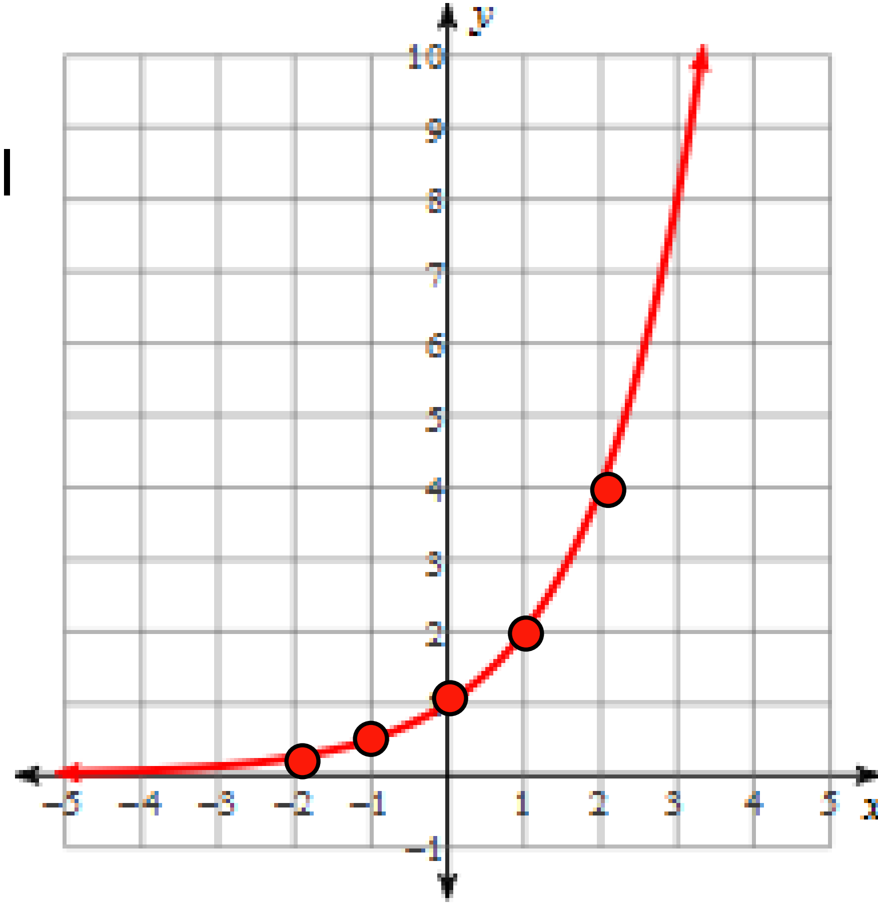
x	$2^{(\ )}$	y
-2	2^{-2}	0.25
-1	2^{-1}	0.5
0	2^{0}	1
1	2^{1}	2
2	2^{2}	4

$* 2$

$* 2$

$* 2$

$* 2$



$$\left(\frac{2}{1}\right)^{-2} = \left(\frac{1}{2}\right)^{2} = \frac{1}{4} = 0.25$$

“negative exponent property”

$$2^0 = 1$$

“zero exponent property”

Exponential Function $f(x)=2^x$

Will the ‘y’ value ever reach zero (on the *left end* of the graph)?

As the denominator gets bigger and bigger, the decimal version of the fraction gets smaller and smaller.

x	$2^{(\ )}$	y
-1	$2^{(-1)}$	$1/2$
-2	$2^{(-2)}$	$1/4$
-3	$2^{(-3)}$	$1/8$
-4	$2^{(-4)}$	$1/16$
-5	$2^{(-5)}$	$1/32$

$f(-1)=1/2$

$f(-2)=1/4$

$f(-3)=1/8$

$f(-4)=1/16$

$f(-5)=1/32$

‘y’ gets closer and closer to zero but never reaches zero.

What is the equation?

x-values increment by '1' each time.

x	f(x)
-2	9
-1	3
0	1
1	0.333
2	0.111
3	0.037
4	0.0124

⅓

Make a ratio: "next over previous"

$$\frac{f(x+1)}{f(x)} = \frac{next}{previous} = \frac{1}{3}$$

$$\frac{next}{previous} = \frac{3}{9}$$

This number is the "growth factor"

The "growth factor" is the base of the exponential.

$$f(x) = \left(\tfrac{1}{3}\right)^x$$

In what portion of the table is it easier to find the growth factor?

Between integer output values.

What is the equation? x-values increment by one each time.

x	f(x)
-2	0.0625
-1	0.25
0	1
1	4
2	16
3	64
4	256

Make a ratio: "next over previous"

$$\frac{next}{previous} = \frac{16}{4}$$

$$\frac{next}{previous} = \frac{4}{1}$$

This number is the "growth factor"

$$f(x) = 4^x$$

Horizontal Asymptote: a horizontal line the graph approaches but never reaches.

$$f(x) = 2^x$$

$y = 0$

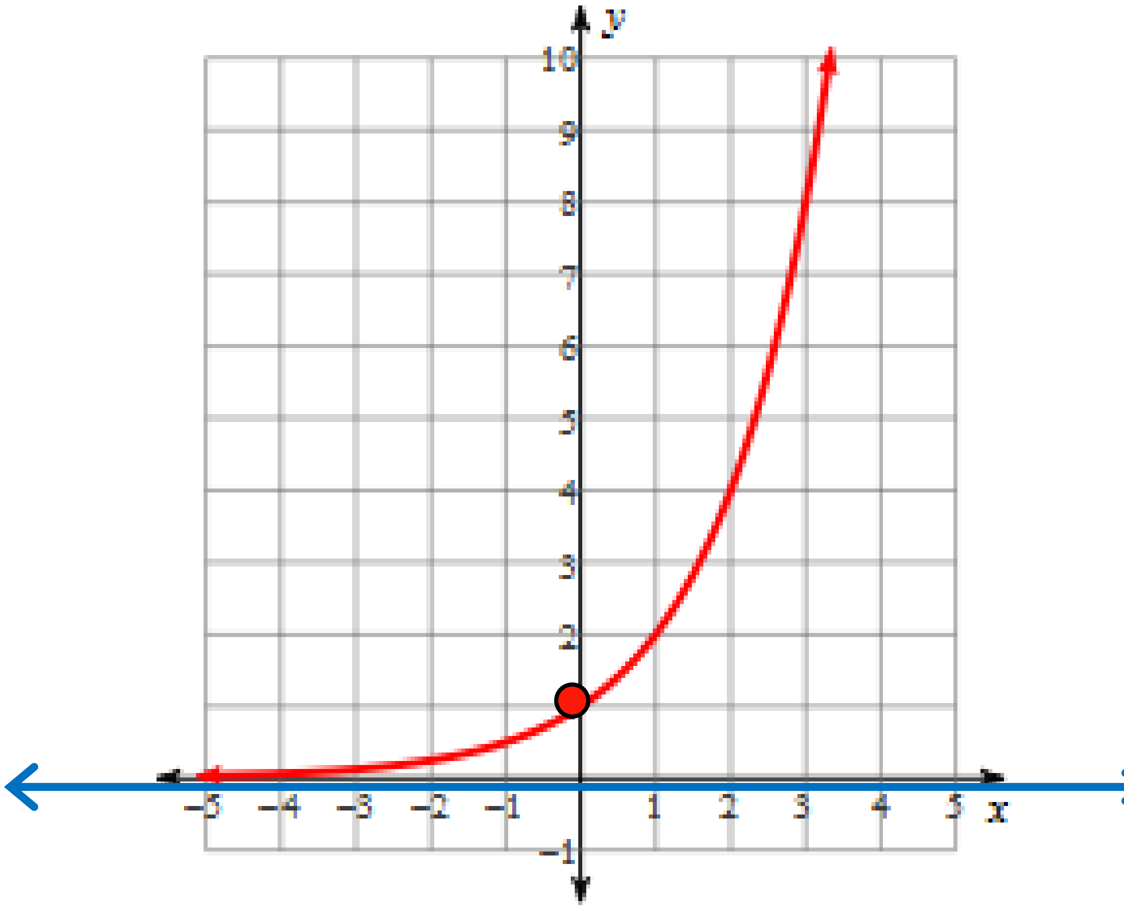
$Domain = ?$

$x = (-\infty, \infty)$

$range = ?$

$y = (0, \infty)$

Parentheses on left end of the interval → range *does include y = 0*



y-intercept = ?

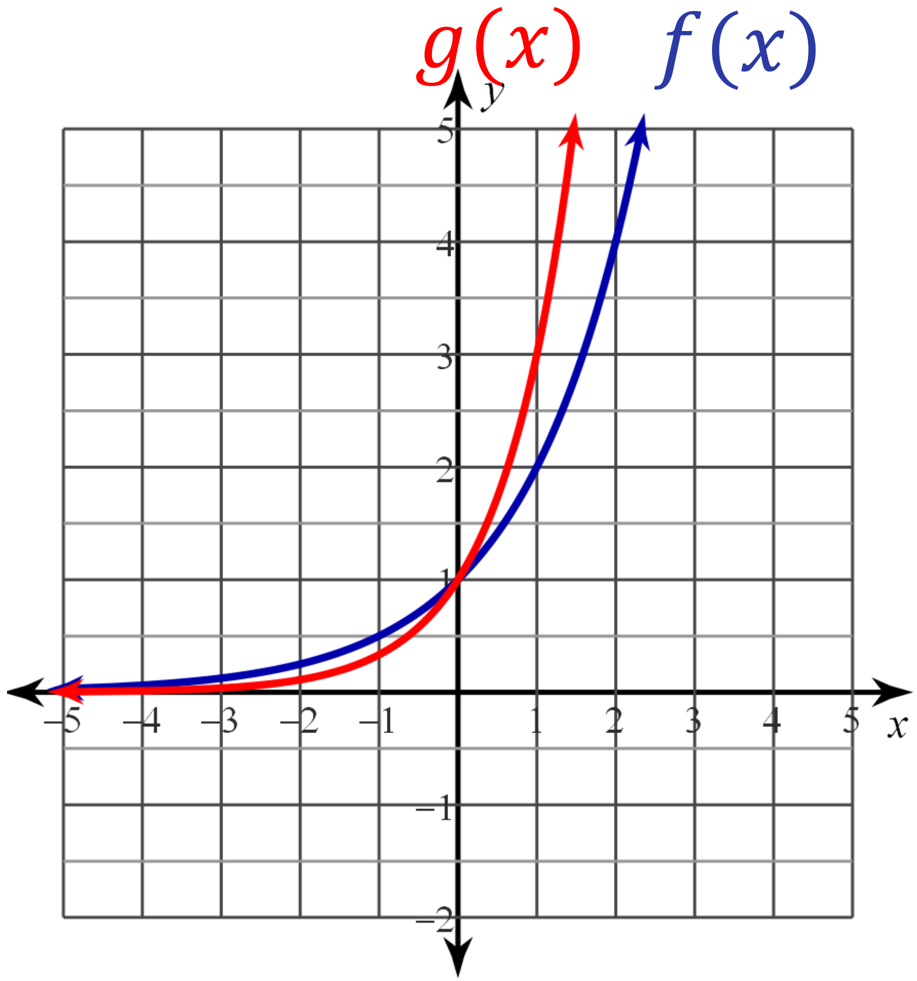
$\boxed{f(0) = y \ \ intercept}$ $\quad f(0) = 2^0 = 1$

Exponential Growth: the graph is increasing (as you go from left to right the graph goes upward). Growth occurs when the base of the exponential is greater than 1.

$$y = b^x$$

$\boxed{\text{‘b’} > 1 \rightarrow \text{growth}}$

$f(x) = 2^x$ $\quad g(x) = 3^x$



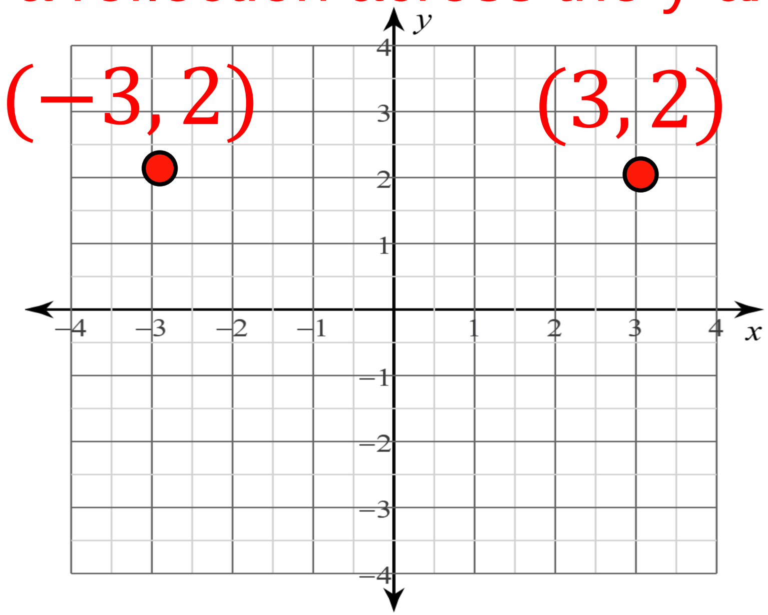
Why do both graphs have the same y-intercept?

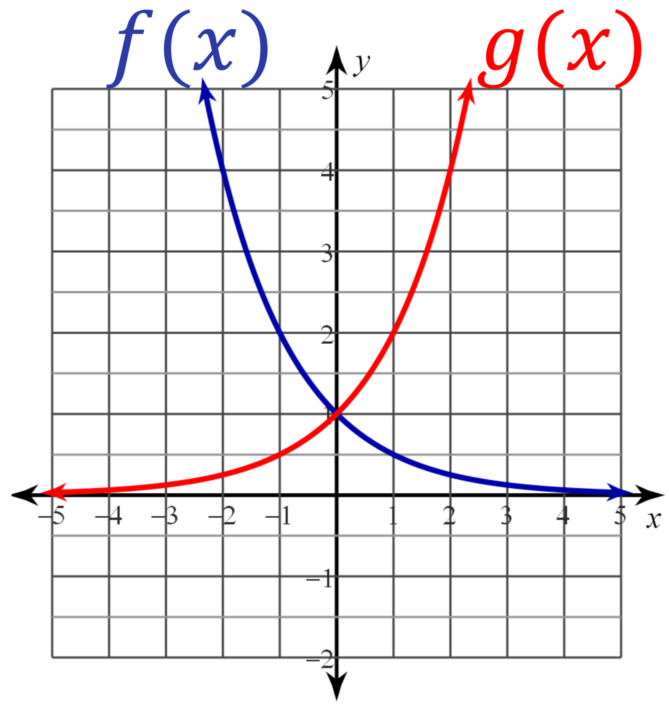
$$f(0) = 2^0 = 1$$

$$g(0) = 3^0 = 1$$

All exponential "*parent functions*" have (0, 1) as the y-intercept.

→ Replacing ‘x’ with ‘(-x)’ causes a reflection across the y-axis





$g(x) = 2^x$ → Reflect across the y-axis

$f(x) = 2^{-x}$

$f(x) = (2^{-1})^x$

Exponent of a Power Property of Exponents

$f(x) = \left(\frac{1}{2}\right)^x$

Negative Exponent Property of Exponents

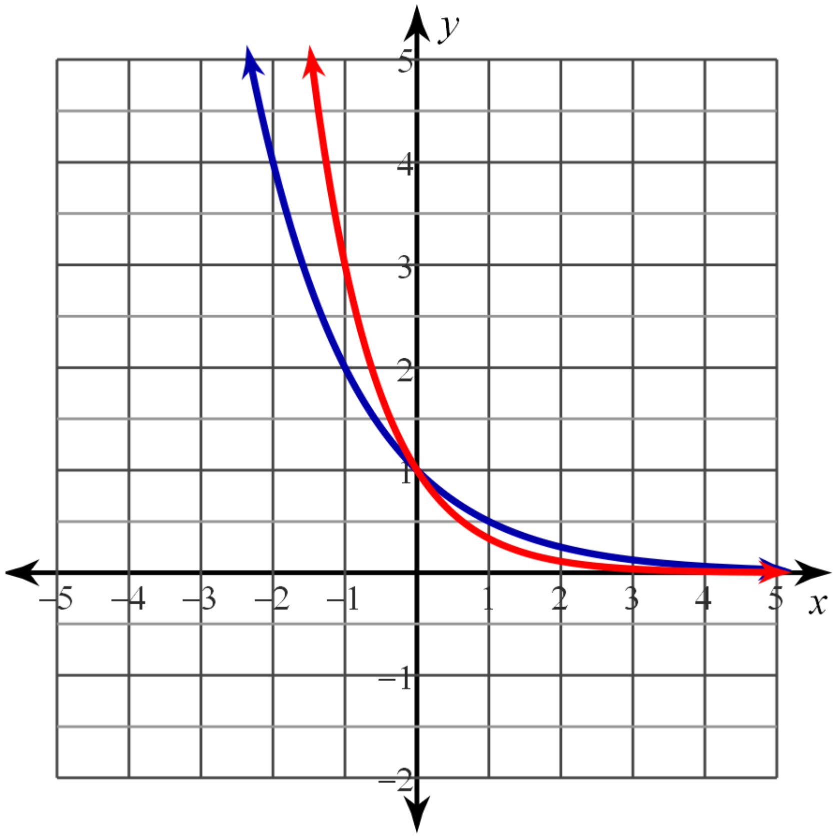
Exponential Decay: the graph is decreasing (as you go from left to right the graph goes downward). This occurs when the base of the exponential is between 0 and 1.

$$y = b^x$$

$0 <$ 'b' $< 1 \rightarrow$ decay

$$f(x) = \left(\frac{1}{2}\right)^x$$

$$g(x) = \left(\frac{1}{3}\right)^x$$



$f(x) = b^x$ Can the base be <u>zero</u>? $\boxed{b \neq 0}$

$g(x) = (0)^x$

x	y
-1	$1/0 = ??$
0	???
1	0

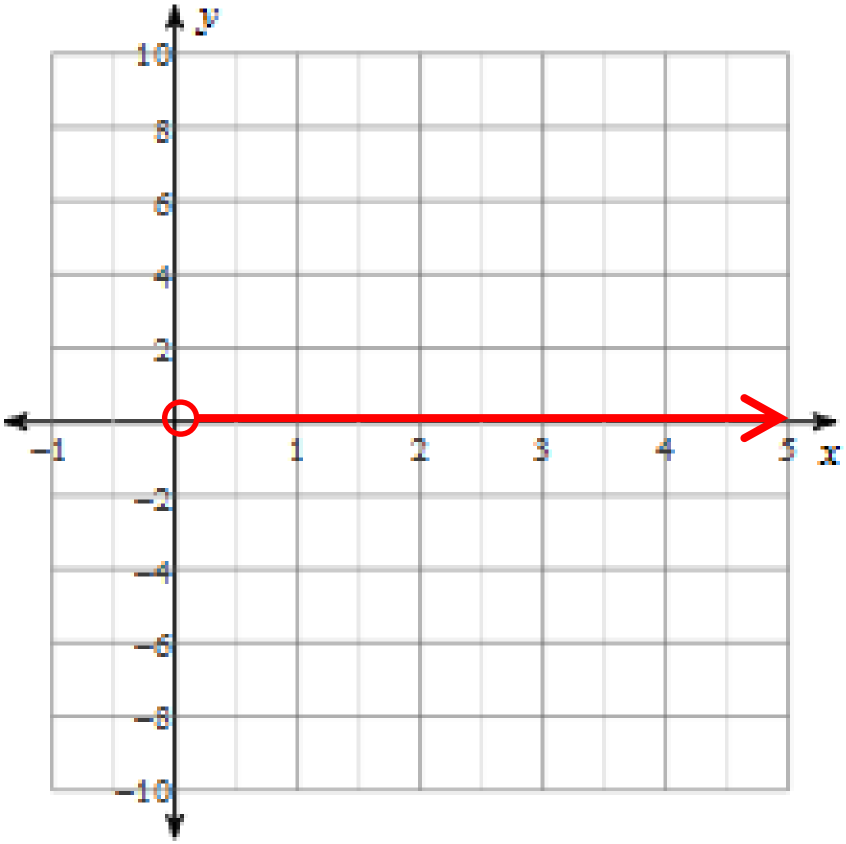
$g(-1) = \left(\frac{0}{1}\right)^{-1}$

$g(-1) = \left(\frac{1}{0}\right)^{1}$

$g(0) = (0)^0$

$= 0? \quad = 1?$

$g(1) = (0)^1$



→ negative number input values are not "mapped" to an output value.

→ input value "0" has an ambiguous output value

→ Any positive number input value is "mapped" to "0"

Can the 'base' be negative?

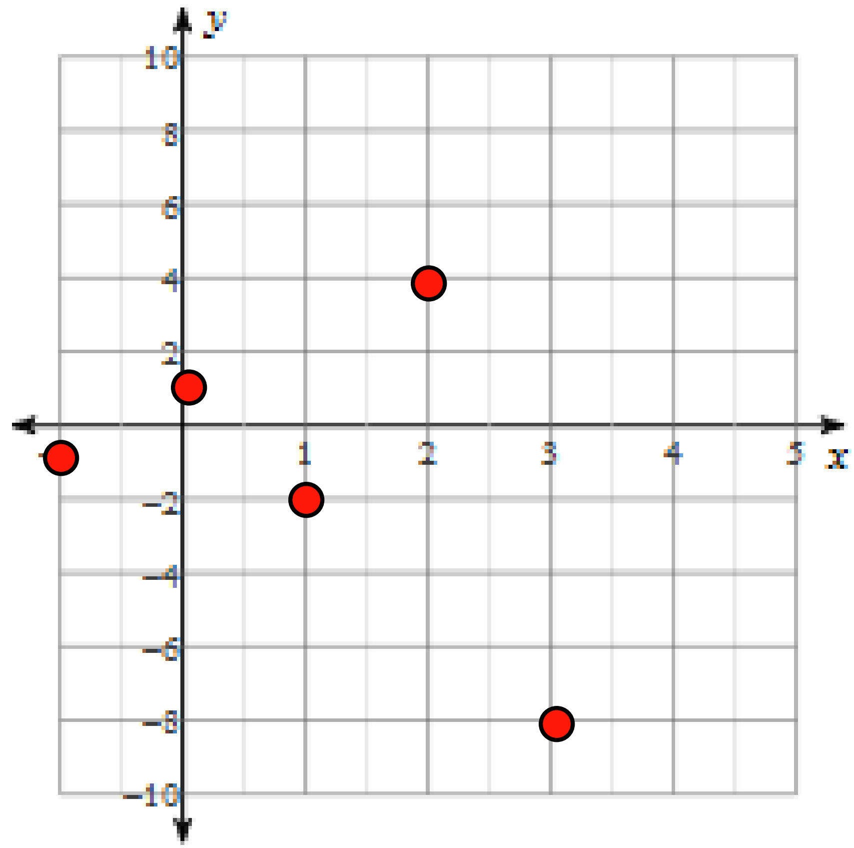
$$f(x) = ab^x$$

'b' > 1 → growth

0 < 'b' < 1 → decay

$$g(x) = (-2)^x$$

x	y	
-1	-0.5	$(-2)^{-1}$
0	1	$(-2)^{0}$
$\frac{1}{2}$	i = ?	$(-2)^{\frac{1}{2}} = \sqrt{-2}$
1	-2	$(-2)^{1}$
2	4	$(-2)^{2}$
3	-8	$(-2)^{3}$



b ≠ negative numbers

Can the base be 1?

$$f(x) = ab^x$$

$$g(x) = (1)^x$$

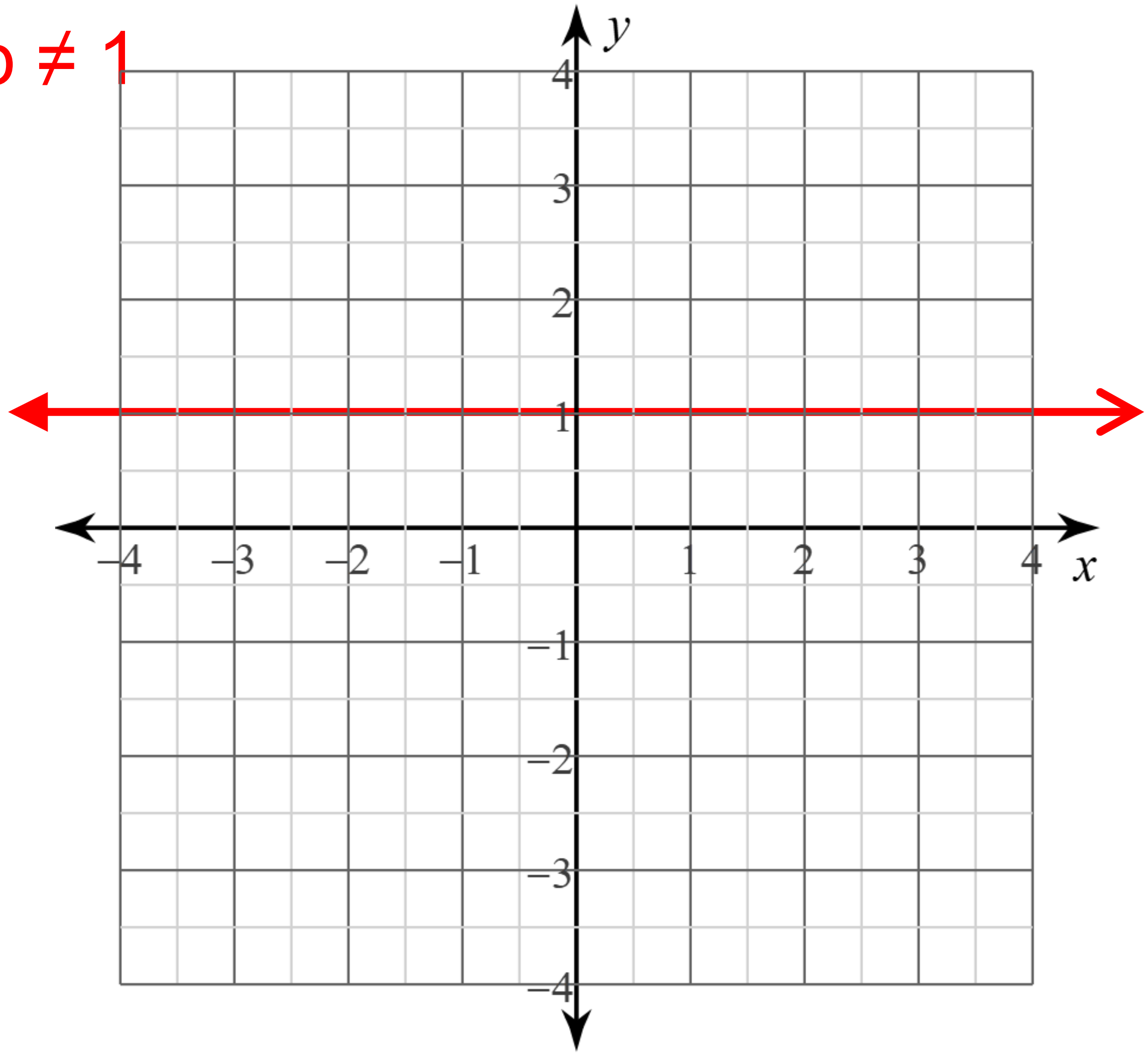
b ≠ 1

x	y
-1	1
0	1
1	1

$(1)^{-1}$

$(1)^0$

1^1



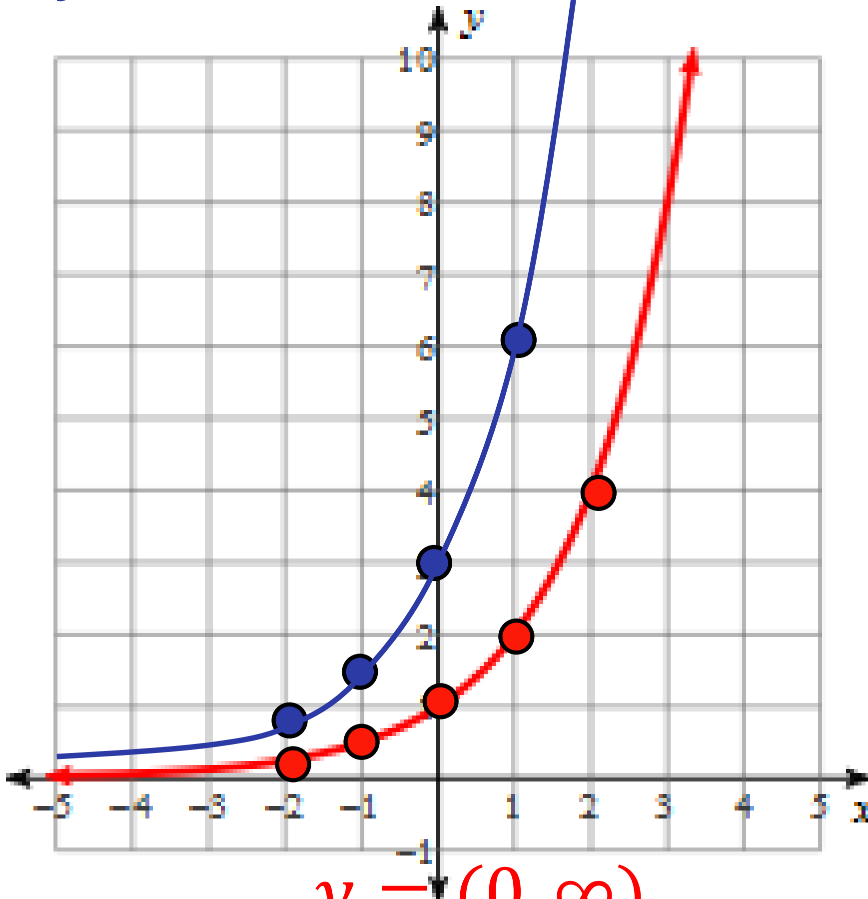
$0 < b < 1,\ \text{OR}\ \ \text{b} > 1$

$b = (0,1) \cup (1, \infty)$

$f(x) = 2^x$ $\quad g(x) = 3(2)^x$

<u>Vertically stretched by a factor of 3</u>

x	$2^{(\ )}$	f(x)	$g(x)$
-2	2^{-2}	0.25	0.75
-1	2^{-1}	0.5	1.5
0	2^{0}	1	3
1	2^{1}	2	6
2	2^{2}	4	12



Horizontal asymptote: $y = 0$

$y = 0$

$range = ?$ $\quad y = (0, \infty)$

$y = (0, \infty)$

$Domain = ?$ $\quad x = (-\infty, \infty)$

$x = (-\infty, \infty)$

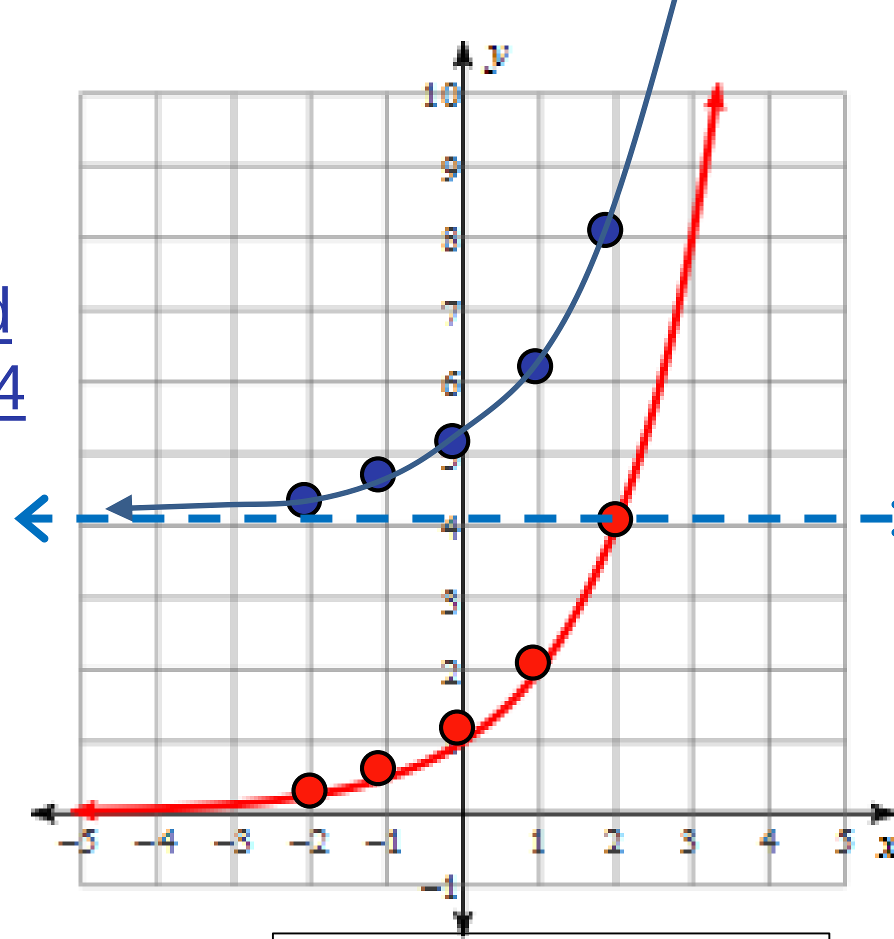
y-intercept = ? $\boxed{f(x): (0, 1)}$

$\boxed{g(x): \ (0, 3)}$

$f(x) = 2^x$ $\quad$ $k(x) = 2^x + 4$

x	$2^{(\ )}$	f(x)	$k(x)$
-2	2^{-2}	0.25	4.25
-1	2^{-1}	0.5	4.5
0	2^0	1	5
1	2^1	2	6
2	2^2	4	8

Shifted UP by 4



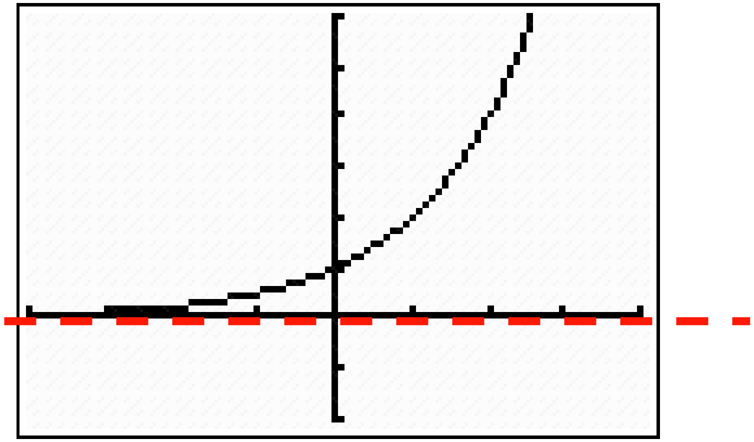
Horizontal asymptote: $\boxed{f(x):\ y = 0}$ $\boxed{g(x):\ y = 4}$

$Domain = ?$ $x = (-\infty, \infty)$ $x = (-\infty, \infty)$

$range = ?$ $\boxed{f(x):\ y = (0, \infty)}$ $\boxed{g(x):\ y = (4, \infty)}$

y-intercept = ? $\boxed{f(x):\ (0, 1)}$ $\boxed{g(x):\ (0, 5)}$

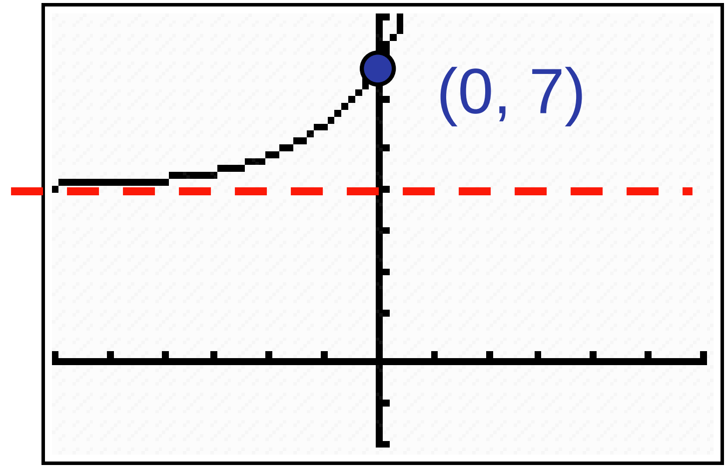
Transformations of the Exponential Function



$f(x) = 2^x$ Base-2 Exponential Parent Function

VSF= 3 Up 4 shift

$$h(x) = 3(2)^x + 4$$



<u>Transformation Form of the Exponential Function</u>

$$y = ab^x + k$$

VSF

vertical shift and horizontal Asymptote

<u>Growth Factor</u> (the base of the exponential)

y-intercept: (0, a + k)

$$h(0) = 3(2)^0 + 4$$

$$h(0) = 7$$

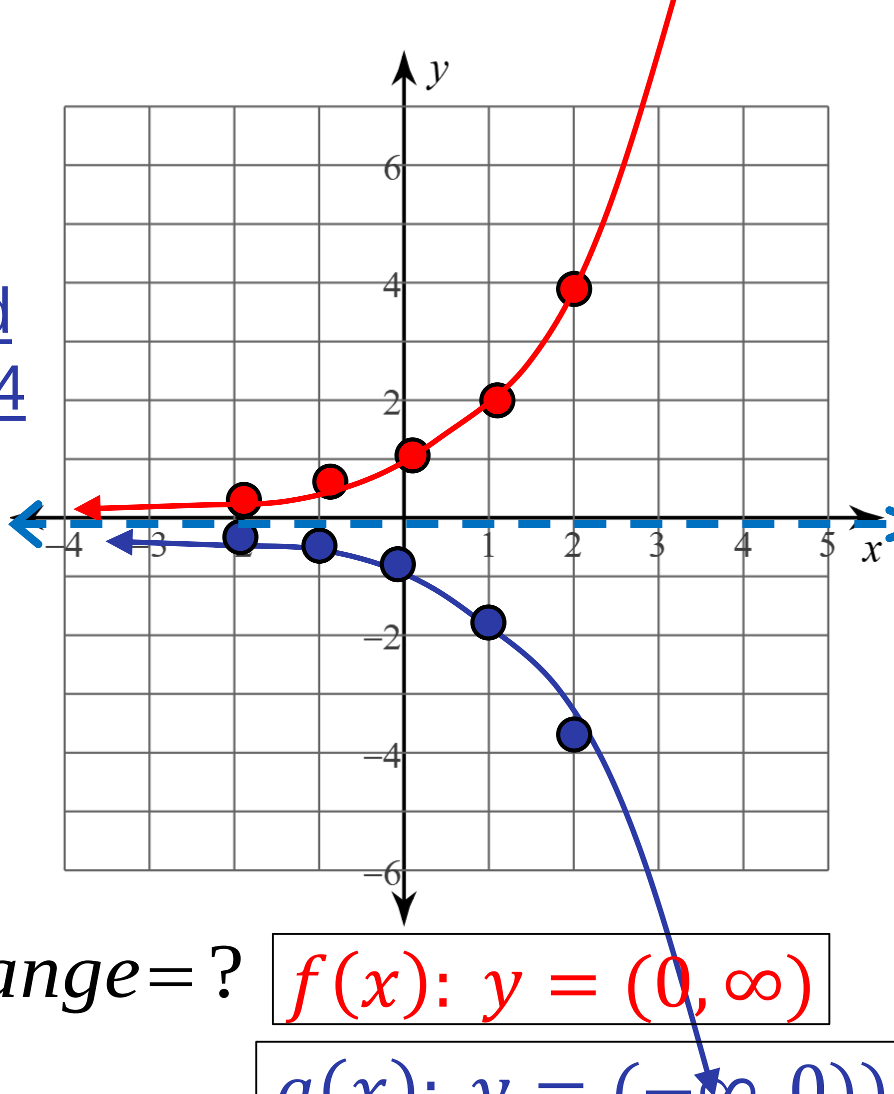
Horiz. asympt: y = k)

y = 4

$f(x) = 2^x$ $\quad$ $\mathrm{k}(x) = -(2)^x$

x	$2^{(\ )}$	f(x)	$k(x)$
-2	2^{-2}	0.25	-0.25
-1	2^{-1}	0.5	-0.5
0	2^0	1	-1
1	2^1	2	-2
2	2^2	4	-4

Shifted UP by 4



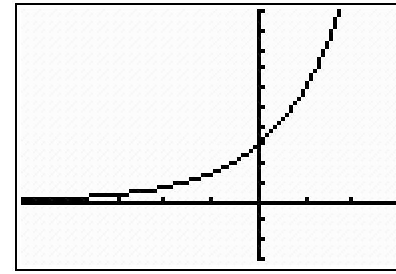
Horizontal asymptote: $\boxed{f(x):\ y = 0}$ $\boxed{g(x):\ y = 0}$

$Domain = ?$ $x = (-\infty, \infty)$ $x = (-\infty, \infty)$

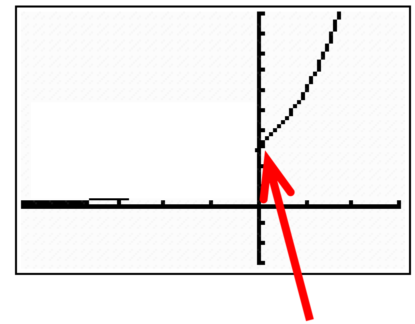
$range = ?$ $\boxed{f(x):\ y = (0, \infty)}$ $\boxed{g(x):\ y = (-\infty, 0))}$

y-intercept = ? $\boxed{f(x):\ (0, 1)}$ $\boxed{g(x):\ (0, -1)}$

<u>Initial Value</u>: (of the exponential) is the <u>vertical stretch factor</u> (*for problems with no up/down shifts)*



If in input is time ("stopwatch time") the <u>initial value</u> occurs when t = 0.



$$f(t) = 3(2)^t$$

Domain: x = [0, ∞)

$$f(0) = 3(2)^0 = ?$$

$$f(0) = 3$$

What is the initial value of: $f(t) = 0.5(3)^t$

Solving a Linear Equation:

"Isolate the variable"

What does "solve" a single variable equation mean?

Find the value of 'x' that makes the equation true.

$$3x + 2 = 4x - 5$$

subtract <u>3x</u> from both sides

$$2 = x - 5$$

add <u>5</u> to both sides

$$7 = x$$

Solving an Exponential Equation: The easiest problem

$$2^x = 2^{4-x}$$

Exponents have to be equal to each other!

$$x = 4 - x$$

+x +x

$$2x = 4$$

÷2 ÷2

$$\boxed{x = 2}$$

Check your answer!

$$2^2 = 2^{4-2}$$

$$7^{2x+1} = 7^{13-4x}$$

$$2x + 1 = 13 - 4x$$

+4x +4x

$$6x + 1 = 13$$

-1 -1

$$6x = 12$$

$$\boxed{x = 2}$$

Equivalent Powers with *different bases.*

$$4^1 = 2^2$$

$$8^1 = 2^3$$

Easy

Harder

$$4^2 = (2^2)^2 \quad = 2^4$$

$$9^2 = (3^2)^2 \quad = 3^4$$

$$9^{2x} = (3^2)^{2x} = 3^{4x}$$

Change the base of the power as indicated:

$$27^1 = 3^?$$

$$16^2 = 4^?$$

$$25^{2x} = 5^?$$

Solving using "convert to same base"

$$2^{4x-1} = 8^{x-1}$$

"convert to same base"

$$2^{4x-1} = (2^3)^{\mathrm{x}-1}$$

Exponent of a power
Exponent Property

$$2^{4x-1} = 2^{3x-3}$$

$$\boxed{x = -2}$$

$$4x - 1 = 3x - 3$$

-3x -3x

$$x - 1 = -3$$

+1 +1

Check your answer!

$$2^{4(-2)-1} = 8^{-2-1}$$

$$2^{-9} = 8^{-3}$$

$$\left(\ 2^{-9} = 8^{-3}\ \right)^{-1}$$

$$2^9 = 8^3$$

$$512 = 512$$

Solving using "convert to same base"

$$9^{2x} = 27^{x-1}$$

"convert to same base"

$$\left(3^2\right)^{2x} = \left(3^3\right)^{x-1}$$

Power of a power
Exponent Property

$$3^{2*2x} = 3^{3(x-1)}$$

Check your answer!

$$9^{2(-3)} = 27^{-3-1}$$

$$3^{4x} = 3^{3x-3}$$

$$9^{-6} = 27^{-4}$$

$$4x = 3x - 3$$

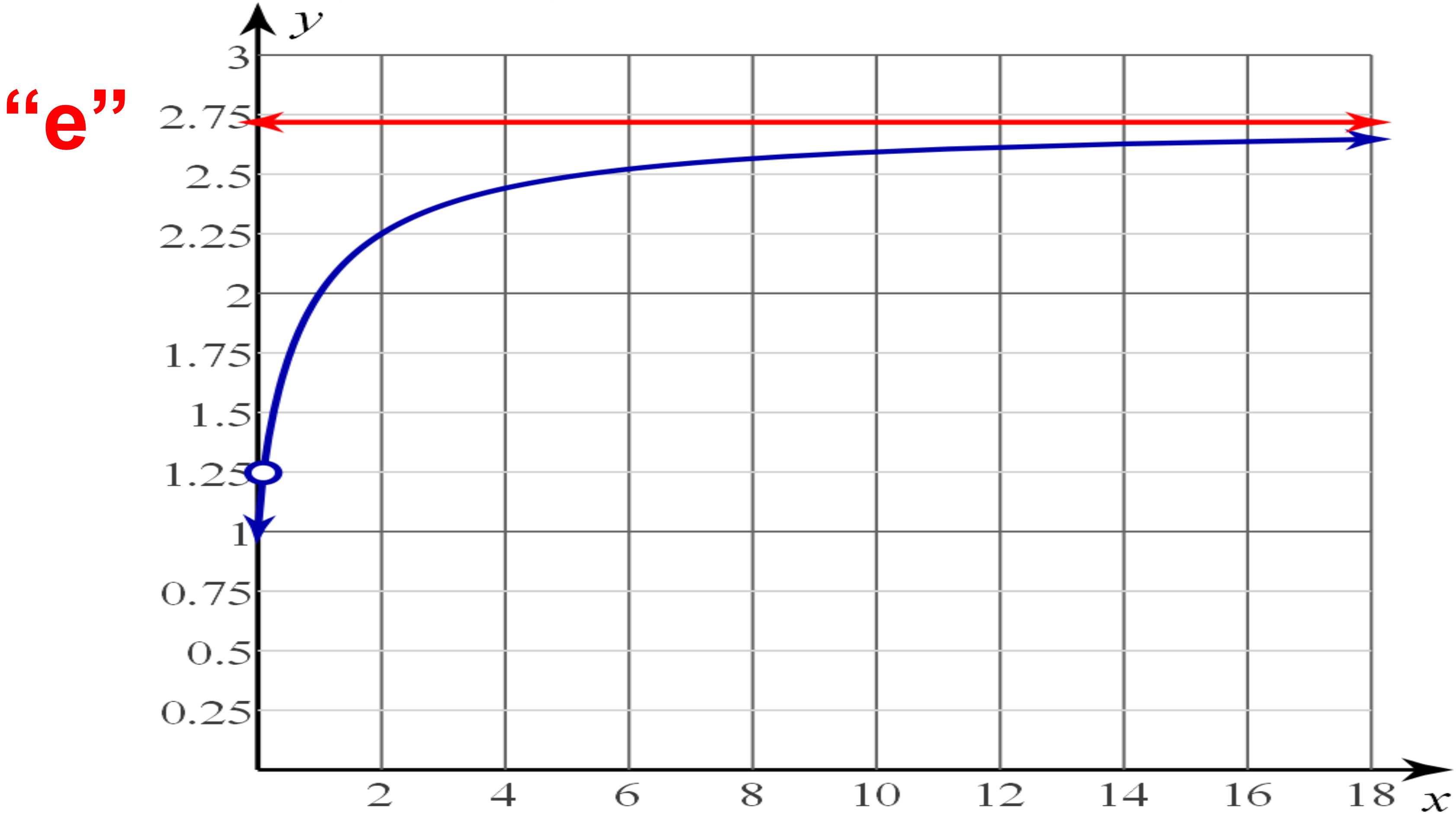
$$\left(9^{-6} = 27^{-4}\right)^{-1}$$

$$\boxed{x = -3}$$

$$9^{6} = 27^{4}$$

$$531441 = 531441$$

$$y = \left(1 + \frac{1}{x}\right)^x$$



$$a_n = \left(1 + \frac{1}{n}\right)^n$$

n	1	2	3	10	40	80	120	200	700	1000
a_n	2.00	2.25	2.37	2.59	2.69	2.70	2.71	2.71	2.72	2.72

$$\lim_{x\to\infty} a_n = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x \approx ? \quad = e$$

$$(e^{2x})^9 = \frac{1}{e^{6x-2}}$$

Negative exponent property

Exponent of a Power Property

$$e^{18x} = e^{-6x+2}$$

Exponents must be equal

$$18x = -6x + 2$$

$$24x = 2$$

$$\boxed{x = \frac{1}{12}}$$