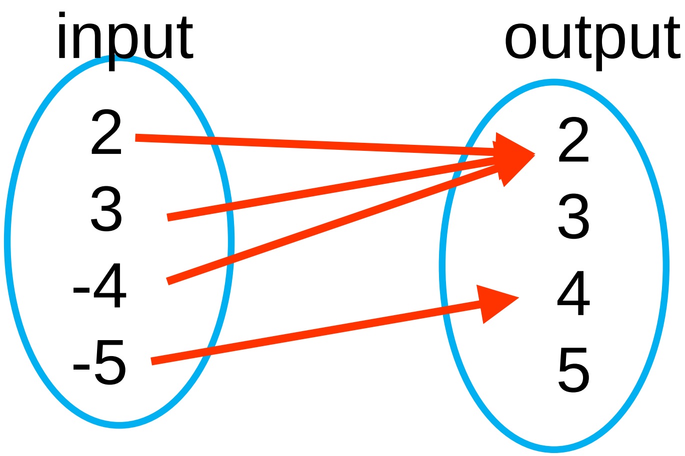
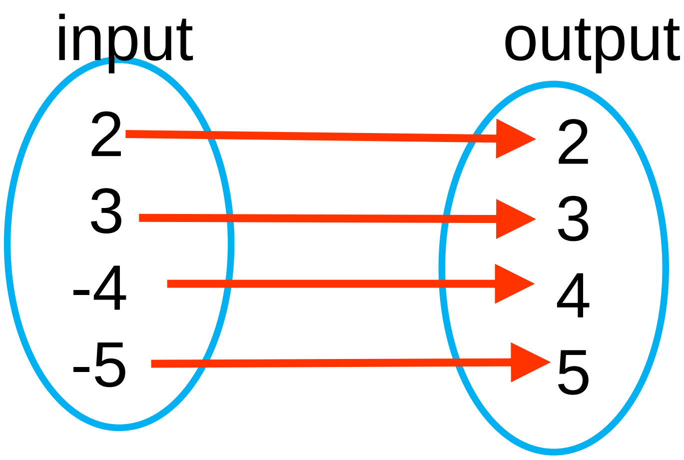
Math-1050
Session #19
Inverse Functions

Function: A relation where each input has exactly one output.



One-to-One Function: Each input has exactly one output, and each output has exactly one input.



Inverse Relation: A relation that interchanges the input and output values of the original relation.

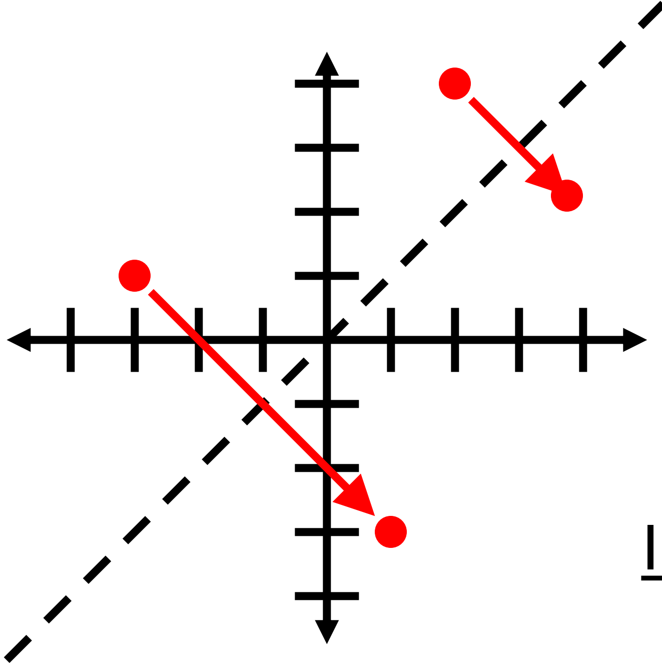
Relation: (-2, 5), (5, 6), (-2, 6), (7, 6)

Inverse Relation: (5, -2), (6, 5), (6, -2), (6, 7)

Inverse Relations

(x, y) = (2, 4) → Inverse the two → (4, 2)

(x, y) = (-3, 1) → Inverse the two → (1, -3)



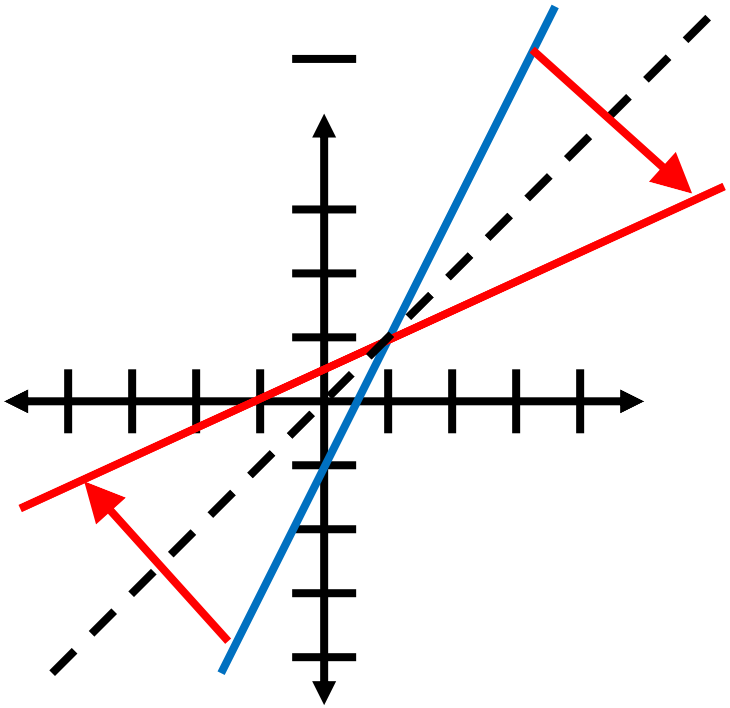
What is the pattern?

A reflection across the line y = x.

<u>Inverse Reflection Principle</u>

Inverse Functions Switch ‘x’ and ‘y’ (then solve for ‘y’)

$$y = 2x - 1 \rightarrow x = 2y - 1 \rightarrow y = \frac{x}{2} + \frac{1}{2}$$



We’re not used to graphing ‘y’ as an input value, then finding the output value ‘x’

So…we can rewrite the equation as ‘y’ in terms of ‘x’ (it’s the same relation).

Bottom line: inverse functions are reflections across the line y = x.

Find the inverse of: f(x) = 4x + 2 Exchange ‘x’ and ‘y’

$$x = 4y + 2$$

This <u>IS</u> the inverse function
(written as: “x as a function of y”)

Rewrite it so that it is written as: “y as a function of x”)

$$x - 2 = 4y$$

subtract ‘2’ (left and right)

$$\frac{x}{4} - \frac{2}{4} = \frac{4y}{4}$$

Divide (all of the) left and right by 4

$$\frac{x}{4} - \frac{1}{2} = y$$

Reduce the fractions

Rearrange into “slope intercept form”

$$y = \frac{x}{4} - \frac{1}{2}$$

This is the inverse of: y = 4x + 2

Function Notation: “the inverse of f(x)”

$$f(x) \qquad\qquad f^{-1}(x)$$

$f^{-1}(x)$ means “the inverse of f(x)”

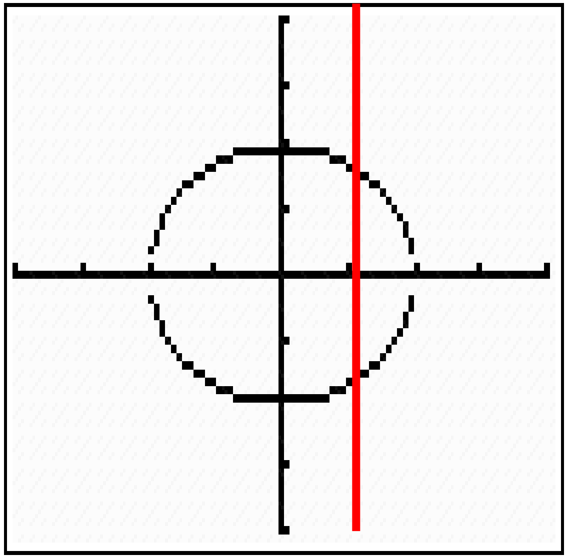
Do not confuse this notation with the negative exponent property:

$$x^{-1} = \frac{1}{x^1}$$

Negative exponent on a number or an expression means “flip the number” (the reciprocal of the number)”

The inverse of a function means “exchange ‘x’ and ‘y’ (then solve for ‘y’).”

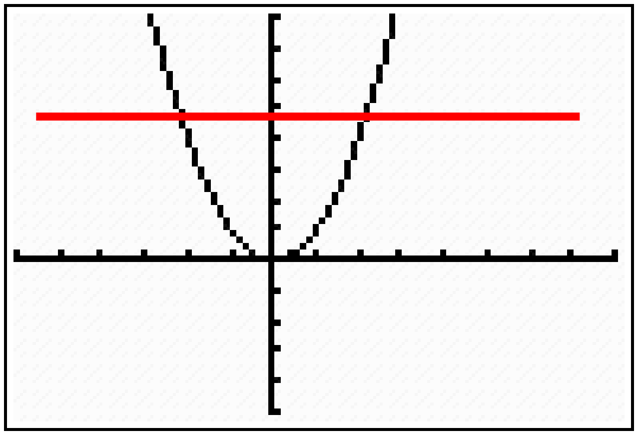
If you have the graph of a relation; <u>how can you tell</u> if the relation is a function?



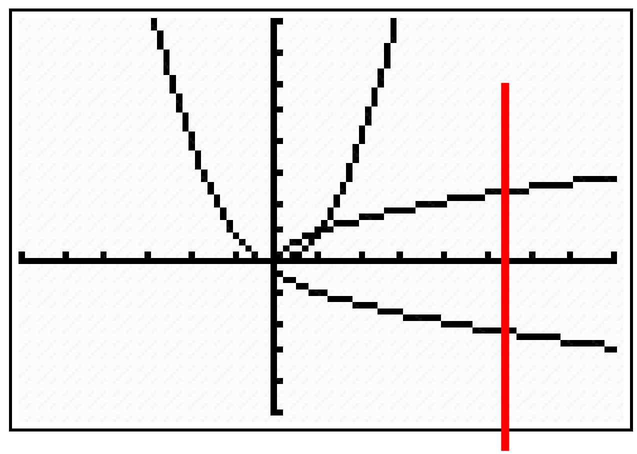
<u>Vertical Line Test</u> if the line intersects the graph more than once, it is <u>NOT</u> a function.

If you have a graph; <u>how can you tell</u> if the function is one-to-one?

$$f(x) = x^2$$



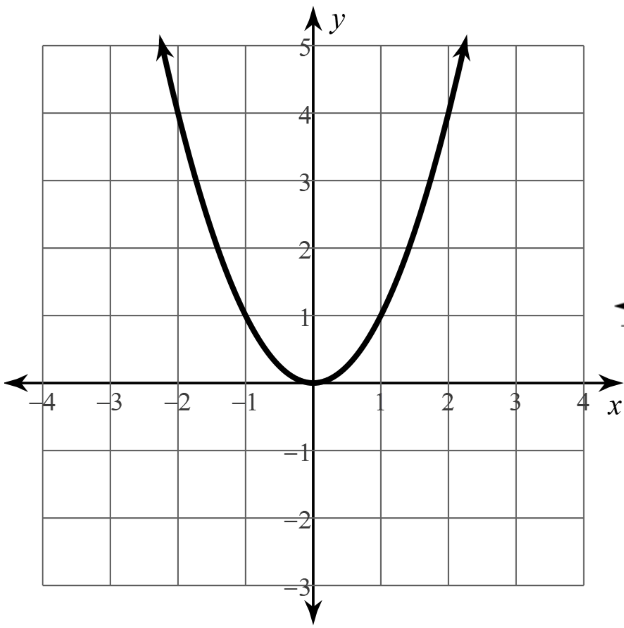
<u>Horizontal Line Test</u>: if the line intersects the graph more than once, then the function is <u>one-two-one</u> and the <u>Inverse</u> of the function is <u>NOT</u> a function.



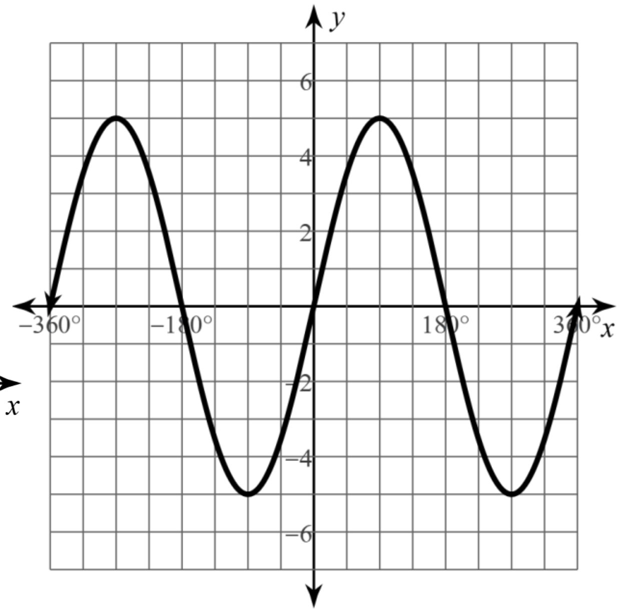
One-to-One functions

For every input there is exactly one output (the definition of a function) AND every output has exactly one input.

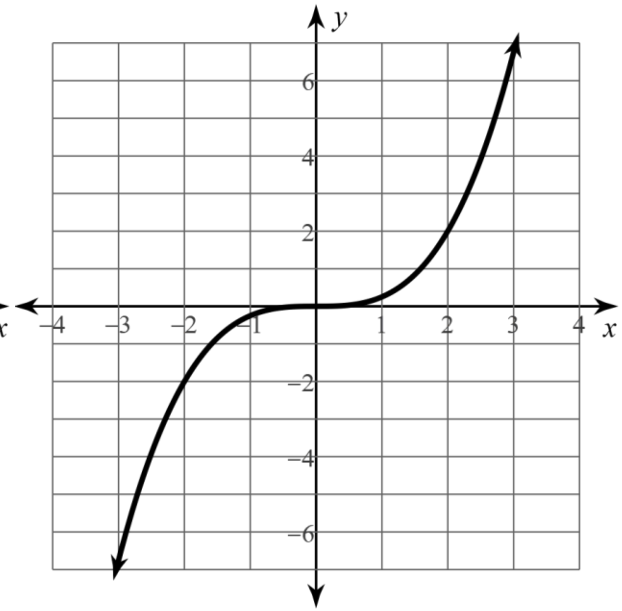
→ It passes both the horizontal and vertical line test.



a.



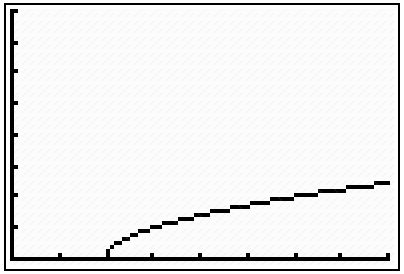
b.



c.

Which function is "one-to-one" ?

$$f(x) = \sqrt{x-2} \qquad f^{-1}(x) = ?$$



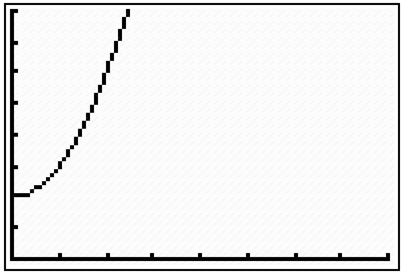
Exchange ‘x’ and ‘y’ in the original relation.

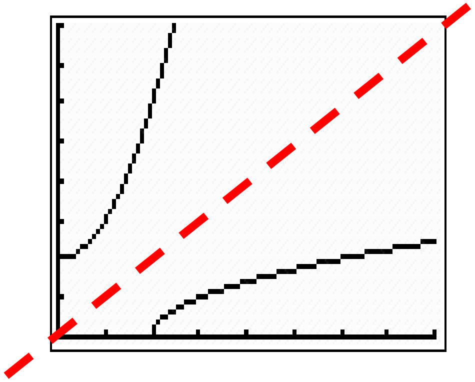
$$x = \sqrt{y-2}$$

This IS the inverse function (written as: “x as a function of y”)

Rewrite it so that it is written as: “y as a function of x”)

$$(x)^2 = \left(\sqrt{y-2}\right)^2 \qquad x^2 = y - 2 \qquad y = x^2 + 2$$





Why do we only graph the right side of the parabola?

Since the x-y pairs of SQRT are all positive, then the x-y pairs of the inverse of the SQRT (square function) will be positive.

Find the inverse of: $f(x) = x^3 - 3$ Exchange 'x' and 'y'

$x = y^3 - 3$

This IS the inverse function, but it is written in the form "x as a function of y"

Rewrite it so that it is written as: "y as a function of x")

$x + 3 = y^3$ Add '3' (left and right)

$\sqrt[3]{x+3} = \sqrt[3]{y^3}$ cubed root both sides

$\sqrt[3]{x+3} = y$ Simplify

$f^{-1}(x) = \sqrt[3]{x+3}$ Is the inverse of: $f(x) = x^3 - 3$

Inverse Function Defined

If f(x) is a one-to-one function with Domain “D” and Range “R” then the inverse function of f(x), denoted

$$f^{-1}(x)$$

Is a function whose Domain is “R” and whose Range Is “D” defined by:

$$f^{-1}(b) = a \quad \text{if and only if} \quad f(a) = b$$

This is just saying the domain of a function is the range of its inverse function.

$$f(x) = \frac{2}{x-3} + 4$$

$$f^{-1}(x) = ?$$

$$x = \frac{2}{y-3} + 4$$

$$y-3 = \frac{2}{(x-4)}$$

$$x-4 = \frac{2}{y-3}$$

$$\boxed{y = \frac{2}{(x-4)} + 3}$$

$$(y-3)(x-4) = 2$$

$$f(x) = \frac{3x}{x+1} + 6$$

$$f^{-1}(x) = ?$$

$$x = \frac{3y}{y+1} + 6$$

$$x - 6 = \frac{3y}{y+1}$$

$$(y+1)(x-6) = 3y$$

multiply this out!

$$xy - 6y + x - 6 = 3y$$

$$xy - 6y - 3y = -x + 6$$

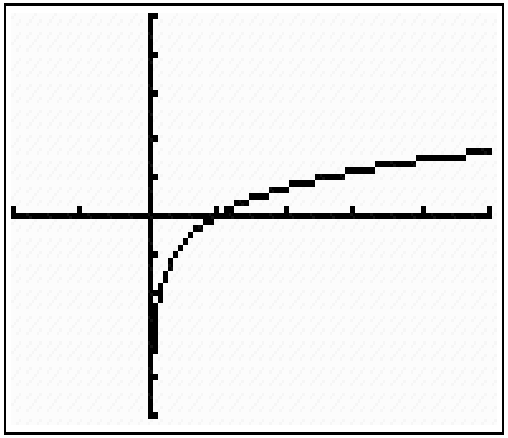
$$xy - 9y = -x + 6$$

$$y(x-9) = -x + 6$$

$$f^{-1}(x) = \frac{-x+6}{(x-9)}$$

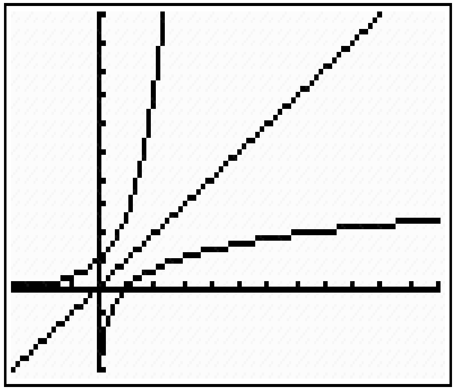
Natural Logarithm Function

$$f(x) = \ln x$$



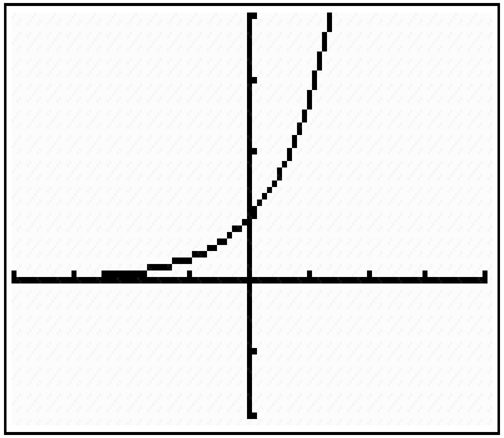
Domain = ?

Range = ?



Exponential Function

$$f(x) = e^{x}$$



Domain = ?

Range = ?

Finding the range of a function can sometimes be difficult.

$$f(x) = \frac{3x - 1}{x + 2}$$

Range of *f(x)* = domain of $f^{-1}(x)$

$$x = \frac{3y - 1}{y + 2}$$

$$x(y + 2) = 3y - 1$$

$$xy + 2x = 3y - 1$$

$$xy - 3y = -2x - 1$$

$$y(x - 3) = -2x - 1$$

$$y = \frac{-2x - 1}{x - 3}$$

$$f^{-1}(x) = \frac{-2x - 1}{x - 3}$$

Range of *f(x)* = domain of $f^{-1}(x)$

Range of *f(x): y ≠ 3*

Verifying the Inverse of One-to-One Functions Algebraically

IF $f(g(x)) = x$ (for every ‘x’ in the domain of g(x))

And

IF $g(f(x)) = x$ (for every ‘x’ in the domain of f(x)

THEN: f(x) is a one-to-one function with inverse g(x)

Verifying if functions are Inverses of each other.

$$f(x) = x^3 + 1 \qquad g(x) = \sqrt[3]{x-1}$$

$$f(g(x)) = \left(\sqrt[3]{x-1}\right)^3 + 1 \qquad = x - 1 + 1 = x$$

$$g(f(x)) = \sqrt[3]{(x^3+1)-1} \qquad = \sqrt[3]{x^3+1-1} = x$$

Verify that the two functions are inverses of each other.

$$f(x)=\frac{x+1}{x} \qquad g(x)=\frac{1}{x-1}$$

Verify that the two functions are inverses of each other.

$$f(x)=\frac{x+2}{3} \qquad g(x)=3x-2$$

Are f(x) and g(x) inverses of each other ?

$$g(x) = \frac{x+1}{4} \qquad f(x) = 4x - 1$$

Are f(x) and g(x) inverses of each other ?

$$g(x) = \frac{(x-1)^2}{5} \qquad f(x) = 1 + \sqrt{5x}$$

The temperature of a bowl of soup is 100 degrees.

Function A: heating by 10 degrees

Function B: cooling by 10 degrees

The temperature of a bowl of soup is 100 degrees. Apply function A then function B (in sequence) to the bowl of soup. What is the final temperature of the soup?

Temperature = 100 + 10 – 10

Composition of *inverse functions*

Function A and Function B are inverses of each other.

Function A: *“does something”* to the input.

Function B: *“undoes whatever function A* did to the input.

2

Function A “does something” to input value 2

Function B “undoes (whatever A did) to the input value 2

What is the output of function B? **2**

What is the inverse function?

1. $f(x) = \{x^4,\ x = [0, \infty)\}$ $\quad f^{-1}(x) = \sqrt[4]{x}$

2. $g(x) = x^{2/3}$ $\quad g^{-1}(x) = x^{3/2}$

3. $h(x) = x^{4/5}$ $\quad h^{-1}(x) = ? \quad = x^{5/4}$

4. $k(x) = x^5$ $\quad k^{-1}(x) = ? \ = x^{1/5} = \sqrt[5]{x}$